

Irrational Exuberance and Herding in Financial Markets

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Abstract

In the context of a two-state, two-trader financial market herd model introduced by Avery and Zemsky (1998) we investigate how informational ambiguity in conjunction with waves of optimism and pessimism affect investor behavior, social learning and price dynamics. Without ambiguity, neither herding nor contrarianism is possible. If there is ambiguity and agents have invariant ambiguity preferences, only contrarianism is possible. If on the other hand ambiguity is high and traders become overly exuberant (or desperate) as the asset price surges (or plummets), we establish that investor herding may drive prices away from fundamentals with economically relevant probability.

Keywords: Social Learning, Herding, Contrarianism, (Partial) Informational Cascade, Ambiguity, Choquet Expected Utility, NEO-Additive Capacities

JEL classification: D81, D82, G12, G14

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1 Introduction

Throughout the past decade financial markets exhibited strong degrees of volatility and were characterized by the formation and subsequent burst of bubbles. The prevalent view in the economic literature is that herding among investors is an important driver for such undesirable market phenomena.

The intuition behind this claim is appealing: Investors face a decision whether or not to buy (or sell) a risky asset. As they observe other investors accumulating on one side of the market, they choose to ignore their own noisy information regarding the asset’s true value and follow the crowd instead. If the crowd is wrong, such herding on the crowd’s action drives prices away from fundamentals contributing towards the formation of bubbles (or excessive downturns). This argument, however, breaks down for most of the existing financial market herd models, see e.g. the seminal works of Avery and Zemsky (1998) and Park and Sabourian (2011). Their models assume that upon the arrival of new information, investors update their beliefs according to Bayes’ Rule and that investor choices are based on subjective expected utility theory, i.e. that ambiguity over probabilities does not matter for financial decision makers. Together, these assumptions essentially prevent the existence of wrong crowds and, thus, wrong herds, compare Eyster and Rabin (2010) and Brunnermeier (2001).¹

In this paper, we want to study how individual investment decisions and the resulting crowd behavior are affected if financial choices are made under ambiguity (Knightian uncertainty). We are particularly interested if ambiguity contributes towards potentially price-distorting herding (contrarianism) and may, thus, support the intuition that investor coordination and bubbles are linked.²

We apply the concept of ambiguity to the two-state, two-trader version of the rational market microstructure herd model of Avery and Zemsky (1998). We assume that investors facing

¹Bayesian updating and preferences in accord with subjective expected utility theory are in line with Barberis and Thaler (2003)’s notion of investor rationality. They argue that the assumption of rationality precludes financial market models from explaining “basic facts about the aggregate stock market, the cross-section of average returns and individual trading behavior” (Barberis and Thaler (2003), p.3).

²A departure from belief updating according to Bayes’ Rule as proposed by e.g. Eyster and Rabin (2010) would also explain the existence of wrong herds. Yet, as Daniel et al. (1998) explicate, any such behavioral bias of investor behavior requires an in-depth experimental and empirical foundation, lest it will be subject to criticism that it is arbitrary. Evidence supports biases such as overconfidence, see Weizsäcker (2010) and Daniel et al. (1998), or probability weightings and loss aversion in line with prospect theory, see Tversky and Kahneman (1992). Applied to herd models these biases cast additional doubt on the rationale that herding may be the cause for market inefficiencies, compare Huber et al. (2015).

ambiguity make decisions in line with non-extreme-outcome-additive (neo-additive) Choquet preferences which were first introduced by Chateauneuf et al. (2007).

The principal model in this paper is related to the one proposed by Ford et al. (2013).³ Yet, we modify and extend their framework in many important ways. First and most importantly, in our framework investor preferences are part of the common knowledge structure of the model. Second, in line with Brunnermeier (2001) and the bulk of the theoretical herding literature, we define herding (contrarianism) as a *switch* in an agent's opinion toward (against) that of the crowd *that has to be induced by the crowd*.⁴ Third, we consider a more general setup as we depart from the General Bayesian Updating (GBU) rule for Choquet preferences proposed by Eichberger et al. (2010). More precisely, we assume that the individual degree of optimism, i.e. investor's ambiguity preference, may vary with the asset price.⁵ Finally, we study a whole class of perturbed versions of our model where the market exhibits marginal uncertainty regarding the true investor preferences. Indeed, Ford et al. (2013)'s assumption that the market is fully ignorant of the true investor preferences can be seen as an extreme special case of the perturbed model setup.

The key insights developed in this paper can be grouped in two categories. First, we characterize conditions under which herding and contrarianism are possible. Second, we discuss how such investor behavior affects market outcomes.

With respect to the first category, we find that informed traders with neo-additive Choquet preferences never herd but show strong contrarianistic tendencies, when beliefs are updated according to the GBU rule.⁶ As we depart from GBU, we specify necessary and sufficient conditions for investor herding. We find that herding becomes possible if high degrees

³Another paper that modifies the model of Avery and Zemsky (1998) to reflect investment choices under ambiguity is the one of Dong et al. (2010). They use smooth ambiguity functions as introduced by Klibanoff et al. (2005) to model ambiguity stemming from multiple priors regarding the distribution of the risky asset. They find that herding is possible if the degree of ambiguity aversion differs between market maker and informed traders. This is in line with Décamps and Lovo (2006), who obtain a similar result for different *risk* preferences among traders and market maker.

⁴Indeed, Ford et al. (2013) do not require that herding and contrarian behavior are crowd-induced.

⁵To motivate this assumption, we appeal to a growing finance literature that assumes that *risk* aversion is subject to change, see for instance Campbell and Cochrane (1999) or Bekaert et al. (2009). Indeed, standard approaches to measure risk aversion via volatility premia, abundantly show that risk aversion depends on market sentiment and recent price trajectories, see e.g. Jurado et al. (2015), Bekaert et al. (2013) and Bollerslev et al. (2011). We posit that if risk aversion is assumed to move with prices, so should ambiguity aversion.

⁶The impossibility of herding derived here contradicts the findings of Ford et al. (2013). This is due to their different definition of herd behavior.

of perceived ambiguity coincide with (potentially irrational) exuberance and despair among informed traders.

Second, in terms of market outcomes, we find that in our two-state, two-trader setup informational cascades occur as soon as investors herd or act as contrarians. Since prices stop moving during an informational cascade we find that both, herding and contrarianism prevent the market from learning the asset's true value. In addition, they have an equal potential to drive prices away from fundamentals. A comprehensive comparative static analysis of the probability of such price distortions is provided.

Informational cascades due to herding and contrarianism, however, exhibit an important qualitative difference, which is revealed by the analysis of the perturbed version of our model. In the perturbed model we still assume that all informed traders have neo-additive Choquet preferences. Yet, market participants think that informed traders have Choquet preferences only with probability $1 - \epsilon$ and that they are expected utility maximizers with probability ϵ . In this case, social learning continues even as investors engage in herd or contrarian behavior.

We find that ambiguity in conjunction with strong exuberance or desperation may cause investors to confidently herd on the wrong state of the world with economically relevant probability in the perturbed model. Markets prone to contrarianism show similar outcomes as in the non-perturbed model, i.e. no learning about the asset's true value and limited long-term price distortions.

The literature of decision making under ambiguity can be grouped in two main approaches that are closely related. First, the multiple prior approach explicitly models a range of probability distributions of the states of the world an individual considers possible (her set of priors) and from which she chooses according to some specified decision rule such as maxmin, compare Gilboa and Schmeidler (1989). Second, the Choquet Expected Utility (CEU) approach models decision making under ambiguity through non-additive probability measures or so-called capacities, compare Schmeidler (1989). If no objective probabilities are available as e.g. in Ellsberg (1961)'s famous mind experiment, CEU agents assign individual likelihoods to different outcomes.

For our application, we choose neo-additive capacities over multiple prior setups as well as general Choquet preferences for three reasons. First, our analysis requires a parametric separation of the degree of perceived ambiguity and the individual attitude towards ambiguity. This makes neo-additive capacities the superior choice when compared to general

capacities, compare Eichberger et al. (2005), Eichberger et al. (2007) and Chateauneuf et al. (2007). Second, focusing on neo-additive capacities is particularly appealing in the Avery and Zemsky (1998) framework, since it rids us of investor beliefs that are unintuitive. For instance, neo-additive beliefs prevent investors from assigning higher likelihoods to the state that is objectively less likely. Finally, in line with Chateauneuf et al. (2007), neo-additive capacities allow us to relate individual degrees of optimism and pessimism to other components of the Avery and Zemsky (1998) model such as the bid and ask price. We can, thus, intuitively describe if and when irrational exuberance (despair) may lead to herding that moves prices away from fundamentals.⁷

We should mention, however, that the theoretical finance literature investigating investor behavior under ambiguity outside social learning settings gravitates towards the multiple prior framework.

Examples of static investment and portfolio choices include the works of Bossaerts et al. (2010), Gollier (2011) and Schröder (2011). They use multiple prior setups such as smooth ambiguity functions as introduced by Klibanoff et al. (2005) (KMM approach) or α -maxmin decision rules to model ambiguity and ambiguity preference.⁸ Intertemporal financial choices under ambiguity are discussed in e.g. Klibanoff et al. (2009). They generalize the KMM framework to an intertemporal setting, deriving a recursive representation for ambiguity preferences. Ju and Miao (2012) employ the generalized KKM framework to model intertemporal asset pricing and investment choices under ambiguity.

There is, however a very strong unifying assumption underlying the KMM, α -maxmin and neo-additive Choquet frameworks. That is, the decision makers' ambiguity attitude is not necessarily limited to aversion but may also reflect lovingness for ambiguous gambles.⁹ Indeed, robustness checks reveal that the results in this paper can be replicated when employing a multiple prior setup with smooth ambiguity preferences or α -maxmin decision rules.

⁷To the best of our knowledge, the concepts of optimism and pessimism have not yet been associated with the mentioned multiple prior frameworks.

⁸The α -maxmin framework is introduced by Ghirardato et al. (2004) and can be seen as the multiple prior counterpart of neo-additive Choquet preferences.

⁹Recent applications of the multiple prior framework include variational and multiplier preferences and are particularly designed to apply ambiguity aversion to intertemporal optimization problems, compare Ghirardato et al. (2004) and Hansen and Sargent (2001) respectively. Since they exclude ambiguity lovingness by definition, they are not suited for our application.

Ambiguity in our framework can be seen as an agent’s lack of confidence in the validity of her information (informational ambiguity). Hence, it is natural to choose a market model, where herding (and contrarianism) is triggered by *information externalities* that an investment decision by one agent imposes on subsequent agents’ expectations about the asset value, compare the seminal work of Bikhchandani et al. (1992).¹⁰

We choose the two-state, two-trader version of Avery and Zemsky (1998) as the baseline model over more recent and complex market microstructure herd models such as Park and Sabourian (2011) or Cipriani and Guarino (2014) since we want to avoid unnecessary distractions due to complex model features.¹¹ Indeed, in the baseline model without ambiguity neither herding nor contrarianism are possible, compare Avery and Zemsky (1998). This constitutes a sharp and, hence, illustrative contrast to investor behavior under ambiguity. Having said that, we will also argue that the insights from this paper are conveniently transferred to more complex setups.

The remainder of this paper is organized as follows: In Section 2, we revisit the model of Avery and Zemsky (1998) and discuss investor behavior if there is no ambiguity. In Section 3, we apply ambiguity to the model of Avery and Zemsky. We derive the necessary and sufficient conditions for herding and contrarianism under ambiguity and discuss corresponding market outcomes in Section 4. In Section 5, we introduce the perturbed model and highlight differences of price-dynamics under herding and contrarianism. Section 6 is devoted to the discussion of the robustness of our findings, while Section 7 concludes. Technical proofs as well as additional material and deep dive analyses are found in the Appendix.

¹⁰Alternative drivers for herd behavior include reputational concerns as well as investigative herding. Reputational herd models modify the agents’ objective functions such that their decisions are affected by positive externalities from a good reputation, see e.g. Scharfstein and Stein (1990), Graham (1999) and Dasgupta et al. (2011). Investigative herd models examine conditions under which investors may choose to base their decisions on the same information resulting in correlated trading behavior, see e.g. Froot et al. (1992) and Hirshleifer et al. (1994).

¹¹Other financial market herd models such as Lee (1998), Chari and Kehoe (2004), and Cipriani and Guarino (2008), investigate how investor herding is related to transaction costs, endogenous timing of trading decisions, and informational spillovers between different assets, respectively.

2 The Baseline Herd Model Without Ambiguity

This section reviews the two-state, two-trader version of the model of Avery and Zemsky (1998) and presents its key property: Without ambiguity no herding and no contrarianism are possible.

2.1 The model setup

Avery and Zemsky (1998) consider a sequential trading model in the spirit of Glosten and Milgrom (1985), consisting of a single asset, informed as well as noise traders and a market maker. The model assumes rational expectations and common knowledge of its structure. Moreover, all decisions in the model are decisions under risk, i.e. there is no ambiguity. We refer to the model specified in this section as the baseline model.

The Asset: There is a single risky asset with unknown fundamental value $V \in \{V_0, V_1\}$, where $V_0 < V_1$. We refer to V_1 as the high state and V_0 as the low state. Without loss of generality, let $V_0 = 0$ and $V_1 = 1$. The prior is fully characterized by the prior probability for the high state $\pi_0 := P(V = V_1)$ and assumed to be non-degenerate, i.e. $0 < \pi_0 < 1$. The asset is traded over T consecutive points in time. After T , the true state of the world is revealed and traders receive their payment accordingly.

The Market Maker: Trading takes place in interaction with a market maker who quotes a bid and an ask price at every time $t = 1, \dots, T$. The market maker only has access to public information, consisting of the history of trades H_t and the risky asset's prior distribution π_0 . The trade history is defined as $H_t := \{(a_1, p_1), \dots, (a_{t-1}, p_{t-1})\}$, where $a_i \in \{\text{buy, sell, hold}\}$ is the action of a trader in period $i \geq 1$ and p_i is the price at which that action is executed.

The relevant public information is fully reflected by the public belief regarding the asset's true value, which is given by $E[V | H_t] = P(V = 1 | H_t) =: \pi_t$.¹² In line with Avery and Zemsky (1998), we also refer to π_t as the asset's price in period t . The market maker is subject to Bertrand competition and, thus, quotes bid and ask prices according to a zero-profit condition. Formally, we have $ask_t = E[V | H_t \cup \{a_t = \text{buy}\}]$ and $bid_t = E[V | H_t \cup \{a_t = \text{sell}\}]$.

¹² π_t uniquely identifies the history of trades up to the number of holds. In particular, it can be bijectively mapped to any order imbalance in the trade history. We will, hence, also refer to π_t as the market's sentiment or degree of optimism. This argument is discussed formally in Section 8.E the Appendix of this paper, see Proposition 8.12.

The Traders: Traders arrive at the market one at a time in a random exogenous order and decide to buy, sell, or not to trade one unit of the asset at the quoted bid and ask prices. Traders are either risk neutral informed traders or noise traders. The fraction of informed traders is denoted by μ . Informed traders base their decision to buy, sell, or not to trade on their expectations regarding the asset's true value. In addition to publicly available information, informed traders form their beliefs based on a private signal $S \in \{S_0, S_1\}$. We refer to S_0 as the low signal and S_1 as the high signal.¹³ Informed traders buy (sell) one unit of the asset if their expected value of the asset $E[V | S, H_t] = P(V = 1 | S, H_t)$ is strictly greater (smaller) than the ask (bid) price quoted by the market maker.¹⁴ Otherwise, they choose not to trade. In contrast to informed traders, noise traders choose their action randomly, that is, they decide to buy, sell, or not to trade with equal probability of $1/3$. Consequently, the probability that a noise trader arrives at the market and either buys, sells or holds the asset is equal to $(1 - \mu)/3$. For notational convenience, we define $(1 - \mu)/3 =: \theta$.

The Private Signal: The distribution of the private signals S_0, S_1 is conditional only on the true state of the world and is denoted by $P(S|V)$. In particular, it does not depend on the trading history H_t . Without loss of generality, we assume symmetric binary signals (SBS) with precision $1 > q > 0.5$, i.e. $P(S_i|V_i) = q$ for $i = 1, 2$. Assuming $q > 0.5$ ensures, that signals are informative in the sense, that they point an informed trader towards the true state of the world. If the low state realizes, then it is more likely to receive a low signal than receiving a high signal (and vice versa if the high state realizes). The larger q , the less noisy and more informative the signal gets.

Updating: Belief updating follows Bayes' rule. Public beliefs are updated from π_t to π_{t+1} when a trading decision a_t is observed in $t + 1$. Similarly, the public belief π_t is updated to a private belief $E[V|S, H_t]$ if a trader arriving at the market at time t has received a private information signal S .¹⁵

The updating rules imply that for any fixed model parameterization the market maker's

¹³Throughout this paper, by an abuse of notation, we also label the informed trader who receives signal S , by S .

¹⁴We can think of traders being endowed with one unit of money. In that sense, selling the asset really means to short-sell it. The investors' endowment is risk and ambiguity free.

¹⁵For the readers convenience, we have stated the formulas for the informed traders' and the market's beliefs as well as bid and ask prices with respect to model parameters in Lemma 8.11 in Section 8.E in the Appendix of this paper.

bid and ask quotes as well as the informed traders' asset valuations in t only depend on the price π_t . We can, thus, view ask_t and bid_t as well as $E[V|S, H_t]$ as functions of π_t . As a notational convention we write $E[V|\cdot, H_t] = E[V|\cdot, \pi_t] = E_{\pi_t}[V|\cdot]$ and $ask_t = ask(\pi_t)$ and $bid_t = bid(\pi_t)$. We will sometimes omit the time index for convenience.

Herding and Contrarianism: In line with Avery and Zemsky (1998), we define herding (contrarianism) as a “history-induced switch of opinion of a certain informed trader in (against) the direction of the crowd”, compare Brunnermeier (2001). For instance, if an informed trader S sells the asset initially based on her asset valuation $E[V|S]$ but decides to buy the asset at $t \geq 1$ after she has observed a price increase (decrease), she is said to engage in buy herding (contrarianism).¹⁶

Informational Cascade: Following Avery and Zemsky (1998) we say that an informational cascade occurs at time t if and only if $P(a_t|V, H_t) = P(a_t|H_t)$, $\forall a_t$. This characterizes a situation where the public cannot or does not infer any information from the observation of a trade, i.e. if $P(V|H_{t+1}) = P(V|H_t)$. To see this, note that during an informational cascade Bayes' Rule implies

$$P(V|H_{t+1}) = P(V|H_t, a_t) = \frac{P(a_t|V, H_t)P(V|H_t)}{P(a_t|H_t)} = P(V|H_t),$$

where the last equality holds due to the informational cascade definition.

Sometimes an informational cascade is also defined as a situation when all informed traders take the same action irrespective of their information signal, compare e.g. Cipriani and Guarino (2008).¹⁷ We note, that as long as the whole model structure is common knowledge, this alternative definition is equivalent to the one we use here. We will, however, also consider a perturbed version of our model, where the market is uncertain regarding the true investor preferences. In this case it is conceivable that all traders take the same action, yet, the market still infers information from observed trades. Hence, the more general definition of Avery and Zemsky (1998) prevents us from wrongly identifying a situation as an informa-

¹⁶The definition for sell herding and contrarianism is symmetric if S buys initially. For formal definitions, see Avery and Zemsky (1998) or Park and Sabourian (2011).

¹⁷The intuition behind this is appealing. If all informed traders take the same action independent of their signal, the market cannot infer any information from their actions any more. Consequently, social learning and price updating stop.

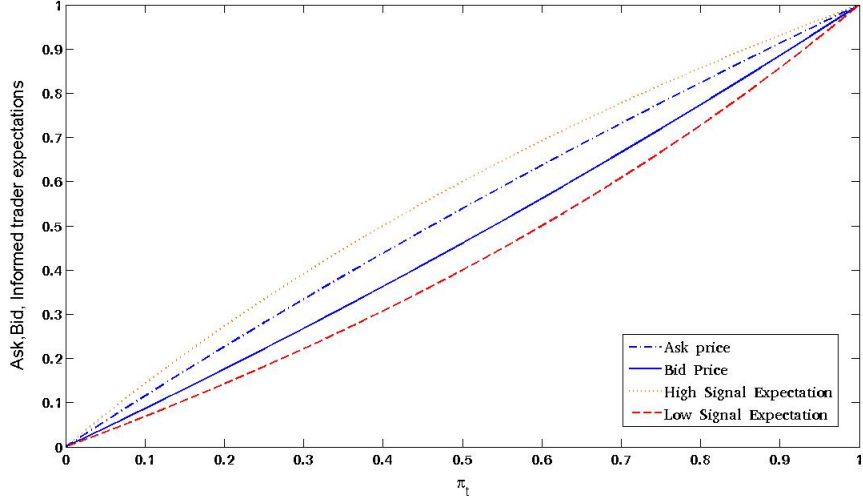


Figure 1: Trading decisions of informed traders in Avery and Zemsky (1998).

Notes: Informed trader expectations $E[V | S, \pi_t]$, bid price bid_t and ask price ask_t are depicted with respect to the public prior belief at time t , π_t . The informed trader share $\mu = 0.3$ and the signal precision $q = 0.6$.

tional cascade, while social learning still continues.¹⁸

Having revisited the two-state, two-trader version of the Avery and Zemsky (1998) model, we now state the key result regarding investor behavior.

2.2 Investor Behavior in the Baseline Model

Informed traders in the Avery and Zemsky (1998) model never change their initial trade decision. Low signals always sell the asset while high signals always buy the asset. This fact is summarized in the following

Proposition 2.1. *Avery And Zemsky*

Informed traders in the two-state, two-trader model of Avery and Zemsky always follow their private signals, i.e. $\forall t$ and histories H_t :

$$0 < E[V | S_0, H_t] < bid_t < \pi_t < ask_t < E[V | S_1, H_t] < 1.$$

¹⁸A formal discussion of the different definitions of informational cascades is provided in Proposition 8.15 in the Appendix.

Proof: Avery and Zemsky (1998). □

An immediate consequence is that neither herding nor contrarianism is possible. Both types of investor behavior require that traders change their initial trade decision, which never happens due to Proposition 2.1. This is illustrated in Figure 1. For any price $\pi_t \in (0; 1)$ (and thus any conceivable history H_t) the expectation of the high signal remains above the ask price while the expectation of the low signal remains below the bid price. Analyses in Avery and Zemsky (1998) and Chamley (2004) show that the market confidently learns about the true value of V in this case. The higher the signal precision q and the informed trader share μ , the faster the market learns.

These clearcut results are an important reason for choosing the two-state, two-trader version of Avery and Zemsky (1998) as our baseline model. It allows us to highlight that introducing informational ambiguity to the model in the next section, indeed, has game changing effects on investor behavior and social learning.

3 Introducing Ambiguity to the Baseline Herd Model

In this section we apply the concept of ambiguity to the model framework of Avery and Zemsky (1998). We show how the assumption that informed traders have non-extreme-outcome-additive (neo-additive) Choquet Preferences affects their asset valuation based on the insights provided by Chateauneuf et al. (2007).¹⁹ We particularly focus our discussion on the role of the perceived ambiguity δ as well as the informed traders' attitude towards ambiguity α . In line with Eichberger et al. (2010), we provide updating rules for the neo-additive Choquet Expected Utility (CEU) beliefs. Finally, formal definitions for herding and contrarianism for investors with CEU preferences are provided. For the remainder of this paper, we refer to this model as the CEU model.

3.1 Investors with NEO-Additive Preferences

As we introduce ambiguity to the model of Avery and Zemsky, we make three general assumptions. First, in order to isolate the effects of ambiguity on investor decisions and social learning, we assume that informed traders as well as the market maker remain risk neutral. Second, the market maker does not perceive ambiguity. We may think of the market maker as an invisible hand that enforces a normatively acceptable price mechanism. Bid and ask prices as well as the public belief π_t should, therefore, be inherently unambiguous. Third, we consider investor preferences to be part of the common knowledge structure of the model.²⁰

To incorporate ambiguity, we assume that informed traders have neo-additive CEU preferences. An individual with this type of preference assigns additive probabilities to every event that does not include the best and the worst outcome. For extreme outcomes neo-additive agents assign a weighted average of additive probabilities and non-additive likelihoods. Since in the two-state world of Avery and Zemsky every outcome is extreme, the resulting neo-additive CEU valuation is particularly simple to derive. In line with Chateauneuf et al. (2007), we infer that an informed trader with neo-additive CEU preferences and signal S

¹⁹A similar exercise has been conducted by Ford et al. (2013). A toolbox of the mathematical objects and results related to the neo-additive ambiguity concept is provided Section 8.G in the Appendix.

²⁰This is a key distinguishing feature from the model of Ford et al. (2013). In Section 5 we relax the common knowledge assumption to study differences of stylized price dynamics in markets prone to herding and markets prone to contrarianism.

values the asset at

$$CEU[V | S, H_t] = (1 - \delta_S)E[V | S, H_t] + \delta_S\alpha, \quad (1)$$

where $\alpha, \delta_S \in [0; 1]$.²¹

$CEU[V | S, H_t]$ is essentially a weighted average of the subjective expected utility (SEU) valuation $E[V | S, H_t]$ and a subjectively assigned likelihood α that $V = 1$ is the true state of nature. In line with Chateauneuf et al. (2007), we regard α as the individual degree of optimism. Indeed, the higher α , the more likely the investor considers the high state to be true, the more optimistic, excited or exuberant she gets regarding the investment prospect and its pay-off (and vice versa).²² The weighting parameter δ_S is the degree of perceived ambiguity and can be viewed as the investor's lack of confidence in her ability to form a SEU belief. The higher δ_S the more the investor relies on her gut feeling α as to whether the low or the high state is true.²³

Throughout this paper we assume that the asset valuations of the different informed trader types are monotone in the sense that $CEU[V | S_0, H_t] \leq CEU[V | S_1, H_t]$. From an economical perspective this can be seen as the ambiguity version of a weak form of the Monotone Likelihood Ratio Property (MLRP) of private signals. Indeed, Park and Sabourian (2011) show that MLRP signal structures imply that the order of the informed traders' asset valuations is the same for all histories H_t . Our monotonicity assumption constitutes a corresponding property under ambiguity. Park and Sabourian (2011) label MLRP and associated trade behavior as “well-behaved”. We presume that this “well-behavedness” is preserved under ambiguity.²⁴

For the remainder of this paper, whenever we speak of CEU , we actually mean CEU with respect to neo-additive capacities unless explicitly stated otherwise.

²¹The same result has been obtained by Ford et al. (2013). In Section 8.G in the Appendix we provide a more detailed and formal derivation of Equation (1).

²²From a decision theoretic perspective, α is primarily a preference parameter describing the investor's attitude towards ambiguity. In line with Ghirardato and Marinacci (2002), the investor is absolutely ambiguity loving (averse) if and only if $\alpha > E[V | S, H_t]$ ($\alpha < E[V | S, H_t]$). She is absolutely ambiguity neutral if and only if $\alpha = E[V | S, H_t]$. Technical details on this are provided in Proposition 8.19 in Section 8.G of the Appendix of this paper.

²³For an intuitive example of an investor facing ambiguity, see Section 8.A in the Appendix.

²⁴We stress that the results of this paper do not hinge on this assumption. Yet, stating the results and discussing them is facilitated.

3.2 Updating CEU Beliefs

3.2.1 General Bayesian Updating (GBU)

The GBU rule of Eichberger et al. (2010) implies that upon the arrival of new information, i.e. the observation of a trade, the additive part of neo-additive beliefs in Equation (1), $E[V | S, H_t]$, is updated according to Bayes' rule as usual. In addition, the degree of ambiguity δ_S is also updated, while the degree of individual optimism α remains fixed.

The updating rule for δ_S is given by

$$\delta_S = \frac{\delta_0}{(1 - \delta_0)P(S|H_t) + \delta_0}. \quad (2)$$

We note again that the dynamics of δ_S solely depend on π_t and that we can, hence, view δ_S as function of π_t .²⁵

The parameter δ_0 can be interpreted as a degree of primary ambiguity that investors perceive when deciding to trade the risky asset. It may stem from the complexity of the asset or from the fact that erratic asset price movements elude established forecasting methods. For instance, derivatives like options, swaps or Collateralized Debt Obligations (CDOs) might trigger a higher primary ambiguity than actual stocks, because they are more difficult to understand and their future values are more difficult to predict accurately. By the same line of reasoning the degree of primary ambiguity should depend on the expertise of the investor. A retail trader perceives much higher degrees of primary ambiguity than a professional investment banker. We assume that $\delta_0 > 0$ is constant across informed traders and during the trading period under consideration $[0; T]$.

Figure 2 illustrates that there are two additional sources of ambiguity that contribute towards the degree of total perceived ambiguity δ_S . It depicts δ_S with respect to the price π_t for informed traders receiving a low signal S_0 as induced by GBU.

In addition to the fixed level of δ_0 , total perceived ambiguity also includes ambiguity stemming from the fact that the private information signal S_0 may contradict the public information reflected in the price π_t and the noisiness of the private signal. For instance, a high price π_t indicates strong market confidence that the high state is the true state. The

²⁵To see this, note that $P(S|H_t) = \pi_t P(S|V_1) + (1 - \pi_t)P(S|V_0)$ by the law of total probability and that $P(S|V_i)$ are time-invariant parameters.

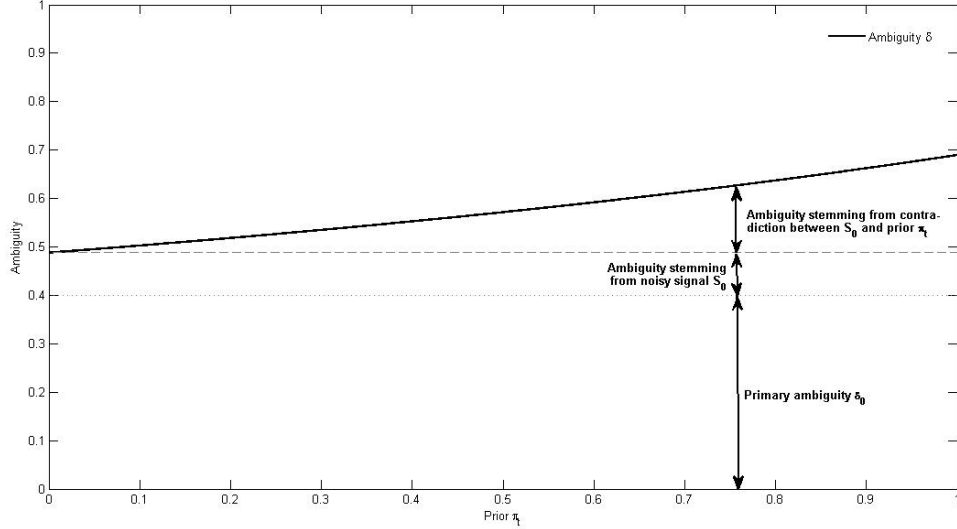


Figure 2: Sources of ambiguity for a low signal

Notes: The graph shows the degree of ambiguity δ_{S_0} with respect to prior π_t for S_0 . The primary ambiguity level is $\delta_0 = 0.4$, the informed trader share is $\mu = 0.3$, the signal precision is $q = 0.7$. δ_{S_0} is computed according to Equation (2).

low signal S_0 , however, suggests that the low state is more likely to be true than the high state, thus contradicting the public information reflected in π_t . Indeed, the greater π_t the more the low signal contradicts the public information and the higher the low signal's total degree of perceived ambiguity δ_{S_0} . Yet, even if public and private information are aligned, i.e. if $\pi_t \rightarrow 0$, the noise in S_0 prevents the informed trader from fully discounting the possibility that the high state is true. As a consequence, the δ_{S_0} remains strictly above the degree of primary ambiguity δ_0 for all prices π_t .²⁶

Eichberger et al. (2010) argue that α is an individual ambiguity preference parameter which should not be affected by the arrival of new information. Yet, since we want to study whether potentially irrational exuberance and outright panics drive investor herding, allowing α to vary might prove insightful. Moreover, the following section shows that the economic literature has readily documented that the conceptually related *risk* preference in fact is subject to change.

²⁶If a trader receives a perfect signal (no noise) there is no informational ambiguity on top of primary ambiguity. In that case, we have $P(S|H_t) \equiv 1$ which implies that $\delta_S \equiv \delta_0$ for all π_t .

3.2.2 Departing from GBU - Varying Degrees of Optimism

It is fairly common in the finance literature to assume that the degree of *risk* aversion depends on the market sentiment, see e.g. Campbell and Cochrane (1999). This is supported by a large body of empirical evidence showing that during crisis periods risk aversion increases, while it tends to vanish during boom phases, see e.g. Jurado et al. (2015), Bekaert et al. (2013) and Bollerslev et al. (2011). Given the conceptual similarities of risk and ambiguity aversion, we argue that the individual degree of optimism α (ambiguity aversion) should also depend on the general market sentiment.

In our model framework market sentiment is best captured by the price π_t . The higher π_t the stronger the degree of optimism exhibited by the market as a whole that V is a lucrative investment opportunity. Since it is reasonable to assume that market-wide optimism affects individual optimism, allowing α to vary with π_t is a feasible generalization of GBU.

Formally, we set $\alpha = \alpha_S(\pi_t \mid q, \pi_0, \cdot)$, i.e. it varies with the price but may also depend on the signal precision, the information signal S or exogenous events. A low signal S_0 , for example, may dampen optimism or boost panic. Likewise, strong and accurate information signals, i.e. a high q , might prevent investors from overreacting to changes in market sentiment, while low q could make the CEU trader particularly susceptible for such mood swings. Finally, we observe that the informed trader share μ plays no explicit role when informed traders form additive beliefs in the baseline model without ambiguity.²⁷ Consequently, we would argue that μ should not directly affect CEU beliefs neither. To ensure this, α must be independent of μ .

We make three additional assumptions regarding $\alpha(\pi_t \mid \cdot)$ for convenience. These assumptions are not crucial for the results derived in this paper, yet, they allow us to state them in a lean and intuitive way.²⁸

(A1) For π_0 both informed trader types act as if they were ambiguity neutral, i.e. $\alpha_S(\pi_0) = E[V \mid S]$.

(A1) implies that S_0 and S_1 type informed traders have different ambiguity functions.

²⁷For example, note that $E[V \mid S_0, H_t] = \frac{(1-q)\pi_t}{(1-q)\pi_t + q(1-\pi_t)}$.

²⁸Section 6 highlights the effects of generalizing (A3) to obtain a framework where investor preferences follow a random distribution. Section 8.B in the Appendix discusses technical effects of dropping (A1) to (A3) on the stated Lemmas and Theorems.

While such an assumption may seem ad hoc, we stress that it is made without loss of generality and only to focus our discussion on the case where the low (high) signal sells (buys) initially. (A1) prevents us from being distracted from less interesting scenarios. For example, it precludes the possibility that the two informed trader types take the same action in $t = 0$, which would cause an informational cascade right at the beginning of trading. Moreover, conditions for herding and contrarianism derived under (A1), i.e. conditions under which traders switch their initial trade decision from selling to buying and vice versa, also hold for weaker forms of switching behavior, i.e. from holding into buying and selling.

(A2) $\alpha(\cdot)$ is sufficiently regular in π_t and the change in α is marginal as the market becomes confident about either state, i.e. $\frac{\partial \alpha}{\partial \pi}(1) = \frac{\partial \alpha}{\partial \pi}(0) = 0$.

(A3) The individual degree of optimism is identical for all low signal traders and all high signal traders respectively.

The updating of δ_S as well as the additive belief component of CEU remains as under GBU. Since δ_S and α_S as well as the additive component $E[V \mid S, H_t]$ can be viewed as functions of π_t , we may also consider CEU as a function of the price. In line with Section 2, we write $CEU[V \mid S, H_t] = CEU_S(\pi)$ for notational convenience.

Now that we have formalized how traders with neo-additive preferences facing ambiguity value the risky asset, we can provide appropriately adjusted definitions for herd and contrarian behavior.

3.3 Herding and Contrarianism in the CEU Model

We modify Avery and Zemsky (1998)'s definition of herding and contrarianism to account for the fact that investor perceive ambiguity and have neo-additive preferences.

Definition 3.1. *Herding With NEO-Additive preferences*

*An informed trader with neo-additive CEU preferences and signal S **buy herds** in \hat{t} at history $H_{\hat{t}}$ if the following three conditions hold:*

*(*BH1) $CEU[V \mid S] < bid_0$, i.e. an informed trader with signal S and neo-additive CEU preferences sells at $t = 0$,*

(*BH2) $CEU[V \mid S, H_{\hat{t}}] > ask_{\hat{t}}$, i.e. an informed trader with signal S and neo-additive CEU preferences buys in $t = \hat{t}$.

(*BH3) $\pi_{\hat{t}} > \pi_0$, i.e. the asset price has increased during $[0; \hat{t}]$.

Analogously, an informed trader **sell herds** in period \hat{t} at history $H_{\hat{t}}$ if and only if (*SH1) $CEU[V \mid S] \geq bid_0$, (*SH2) $CEU[V \mid S, H_{\hat{t}}] < bid_{\hat{t}}$, and (*SH3) $\pi_{\hat{t}} < \pi_0$ hold simultaneously.

These modifications ensure that in line with Brunnermeier (2001) and the bulk of the theoretical herding literature, an informed trader's switch in opinion is still induced by the observed trade history. More precisely, herding would by definition be impossible if the trade decisions of other investors were not observable, compare opaque market in Park and Sabourian (2011).²⁹

The corresponding definition for contrarian behavior is

Definition 3.2. Contrarianism With NEO-Additive preferences

An informed trader with neo-additive preferences and signal S acts as a **buy contrarian** in \hat{t} at history $H_{\hat{t}}$ if the following three conditions hold:

(*BC1) $CEU[V \mid S] < bid_0$, i.e. an informed trader with signal S and neo-additive CEU preferences sells in $t = 0$.

(*BC2) $CEU[V \mid S, H_{\hat{t}}] > ask_{\hat{t}}$, i.e. an informed trader with signal S and neo-additive CEU preferences buys in $t = \hat{t}$.

(*BC3) $\pi_{\hat{t}} < \pi_0$, i.e. the asset price has decreased during $[0; \hat{t}]$.

Analogously, an informed trader acts as a **sell contrarian** in period \hat{t} at history $H_{\hat{t}}$ if and only if (*SC1) $CEU_{\delta_0, \alpha}^S[V] > ask_0$, (*SC2) $CEU_{\delta_0, \alpha}^S[V \mid H_{\hat{t}}] < bid_{\hat{t}}$, and (*SC3) $\pi_{\hat{t}} > \pi_0$ hold simultaneously.

With these definitions at hand, we are now prepared to investigate investor behavior in the model of Avery and Zemsky under ambiguity.

²⁹This is not the case in Ford et al. (2013). They consider any buy (sell) decision of a low (high) CEU signal to be a corresponding herding trade as long as the price has increased (decreased).

4 Investor Behavior and Social Learning under Ambiguity

In this section we present the main results. We will first solve the CEU model by providing equilibrium prices and updating rules. We then investigate investor behavior in the CEU model under GBU and varying α , respectively. As we derive necessary and sufficient conditions for herding and contrarianism, we note that both types of investor behavior will inevitably lead to informational cascades. As we study the characteristics of the corresponding market outcomes, we find that herders and contrarians are equally likely to cause prices to move away from fundamentals. This probability is derived analytically and comparative statics are conducted.

4.1 Solving the CEU Model

We conjecture that unlike in the baseline model without ambiguity in Section 2, informed traders with neo-additive CEU preferences may change their initial trade decision after having observed certain histories of trades. If S_0 and S_1 take the same action at any time t , an informational cascade occurs. Social learning stops and the market maker quotes ask and bid prices equal to π_t .³⁰ Even if there is no informational cascade, any decision change of an informed trader affects the market maker's price setting as well as the public belief updating. We shall begin by formalizing the market maker's price setting.

Lemma 4.1. *Equilibrium Prices in the CEU Model*

Under the assumptions of the CEU model, let bid_t and ask_t be the bid and ask prices that are quoted in the Avery and Zemsky model at any time t . If there is no informational cascade in t , then the market maker quotes

$$bid_t^{CEU} = \min\{\max\{bid_t; CEU[V \mid S_0, H_t]\}; \pi_t\}$$

and

$$ask_t^{CEU} = \max\{\min\{ask_t; CEU[V \mid S_1, H_t]\}; \pi_t\}.$$

If there is an informational cascade in t , then the market maker quotes $bid_t^{CK} = ask_t^{CK} = \pi_t$

Proof: We have already established pricing given that S_0 and S_1 take the same action at t , i.e. if there is an informational cascade. As long as S_0 sells and S_1 buys, prices are as in the

³⁰Compare Avery and Zemsky (1998) and Cipriani and Guarino (2008) for detailed discussions of informational cascades as well as Proposition 8.15 in Section 8.F in the Appendix of this paper.

Avery and Zemsky model. Indeed, since $CEU[V | S_0, H_t] < bid_t < \pi_t$ in this case, we have that $bid_t^{CEU} = bid_t$. The same argument applies for $ask_t^{CEU} = ask_t$.

Moreover, monotonicity of the CEU-beliefs implies, that S_1 never sells if S_0 does not sell, and that S_0 never buys if S_1 does not buy. This leaves only two additional cases to consider. First, the case where S_0 holds and S_1 buys and second, the case where S_1 holds and S_0 sells. For symmetry reasons, we will only prove the first case.

Let ask_t and bid_t denote the ask and bid prices the market maker quotes in the Avery and Zemsky model. Assume that at some time t , the high signal still buys and we have $bid_t \leq CEU[V | S_0, H_t] \leq \pi_t < ask_t$. This implies that the low signal with CEU preferences holds in t .

Since the market maker (and his fictive Bertrand competition) are aware of this, the zero-profit condition implies an increase of the quoted bid price to $bid_t^{CEU} = CEU[V | S_0, H_t]$. If the market maker set $bid_t^{CEU} < CEU[V | S_0, H_t]$, then he would make an average gain on every sell of $\pi_t - bid_t^{CEU}$. The market maker's competition's best response is to quote a bid price $\widetilde{bid}_t^{CEU} > bid_t^{CEU}$ such that $\widetilde{bid}_t^{CEU} < CEU[V | S_0, H_t]$ and $\pi_t - \widetilde{bid}_t^{CEU} > 0$. In other words, the competition can quote a better bid price, thereby drawing away all noise traders that sell the asset, while still making profits. In turn, the optimal response of the market maker then is to increase bid_t^{CEU} correspondingly. This price war continues until $bid_t^{CEU} = \widetilde{bid}_t^{CEU} = CEU[V | S_0, H_t]$. If the market maker quoted $bid_t^{CEU} > CEU[V | S_0, H_t]$, then the low signal would sell at t . Consequently, the market maker would make an average loss of $bid_t - bid_t^{CEU}$ for each sell he fills and would, therefore, eventually go out of business. Since the competition is in the same situation, no market maker has an incentive to deviate from the equilibrium bid price of $CEU[V | S_0, H_t]$.

If $\pi_t < CEU[V | S_0, H_t] \leq ask_t$, then the market maker quotes an equilibrium bid price of $bid_t^{CEU} = \pi_t$. A downward deviation is not possible due to Bertrand competition by the same reasoning as above. An upward deviation would cause the market maker to make average losses of at least $\pi_t - bid_t^{CEU}$ for each sell he fills and, therefore, would again lead to bankruptcy.

□

There are two important implications of Lemma 4.1.

First, as soon as an informed trader switches into holding, the market maker can make

profits. As long S_1 holds and $CEU_{S_1}(\pi)$ remains above the market price π (S_0 holds and $CEU_{S_0}(\pi)$ remains below π), the market maker on average profits from every buy (sell) he fills. This is in line with the intuition that if traders depart from rationality in the sense of Barberis and Thaler (2003), there will be opportunities for other market participants to make money additional money. In all other cases the zero-profit condition holds.

Second, note that the quoted ask (bid) price in the CEU model remains the same as in the Avery and Zemsky model as long as the high signal S_1 buys and the low signal S_0 does not buy (the low signal S_0 sells and the high signal S_1 does not sell). This is crucial for the derivation of the result on the possibility of herding and contrarianism and it ensures that the corresponding results transfer to the perturbed model setup of Section 5.

Next, we derive the updating rules for the public belief π_t in the CEU model.

Lemma 4.2. *Public Belief Updating in the CEU Model*

Under the assumptions of the CEU model, let bid_t and ask_t be the bid and ask prices that are quoted in the Avery and Zemsky model at any time t .

- (1) *If the high signal buys and the low signal sells at time t , then π_{t+1} is as in the Avery and Zemsky model.*
- (2) *If both signals take the same action in t , then there is an informational cascade and $\pi_{t+1} = \pi_t$.*
- (3) *If the high signal buys and the low signal holds at time t , then*

$$\pi_{t+1} = \begin{cases} ask_t, & \text{if } a_t = \{buy\} \\ \pi_t, & \text{if } a_t = \{sell\} \\ bid_t, & \text{if } a_t = \{hold\}. \end{cases} \quad (3)$$

- (4) *If the high signal holds and the low signal sells at time t , then*

$$\pi_{t+1} = \begin{cases} \pi_t, & \text{if } a_t = \{buy\} \\ bid_t, & \text{if } a_t = \{sell\} \\ ask_t, & \text{if } a_t = \{hold\}. \end{cases} \quad (4)$$

Proof: Case (2) is directly implied by the definition of an informational cascade. To see, that cases (1), (3) and (4) hold, note, that informed traders still reveal their fundamentally driven signal through their action. Since in case (1), informed traders decide as in the Avery and Zemsky model price updating also coincides. In case (3) - where S_1 buys and S_0 holds - a sell would contain no informational value, because it is a noise trade for sure. Consequently, the price remains constant. If a buy is observed, the market maker knows, that it is conducted by either the high signal or a noise trader. Consequently, updating after a buy is exactly the same as in the Avery and Zemsky model. Finally, if a hold is observed, the market maker knows, that it is due to the low signal or a noise trader. This is equivalent to the situation of an observed sell in the Avery and Zemsky model. Consequently, the market maker sets $\pi_{t+1} = bid_t$, when observing a hold. The argument is symmetric for case (4), i.e. if S_1 holds and S_0 sells.

□

Lemma 4.2 implies that the probability to observe a price increase (decrease), i.e. $\pi_{t+1} > \pi_t$ ($\pi_{t+1} < \pi_t$) remains constant as long as there is no informational cascade.³¹ This is crucial for deriving a closed formula for the probability of price-distorting market outcomes below.

Having solved the model dynamics, we turn to the analysis of investor behavior in the CEU model. We will first focus on the case where informed traders update their CEU belief in accordance with the GBU rule. We will then consider the general case, where α may vary with the price π_t .

4.2 Investor Behavior in the CEU Model under GBU

This section shows that if informed traders perceive ambiguity and update their neo-additive CEU preferences according to the GBU rule, i.e. they exhibit invariant ambiguity preference α , then there is *no herding* in the CEU model. At the same time, informed traders show *strong contrarian tendencies* that prevent the market from becoming confident about either state.

Theorem 4.3.

³¹In particular, note that the probability of a price increase (decrease) is the same as in the baseline model, where it coincides with the probability of buy (sell), see Lemma 8.11 in Section 8.E in the Appendix of this paper for the respective formulas.

In the CEU with $\alpha \in [0; 1]$ fixed, **no herding** can occur. If, in addition, S_0 sells initially and S_1 buys initially, then **contrarianism occurs** with positive probability.

Proof: Since this proof is fairly technical, it is left to the Appendix. □

We require, that S_0 (S_1) sells (buys) initially to avoid situations in which both informed trader types take the same action initially, thereby causing an informational cascade right at the beginning of trading.³² Note, that Theorem 4.3 contradicts some of the key results in Ford et al. (2013). This is because we apply different definitions for herding and contrarianism. Although the mathematical proof is left to the Appendix, we would like to provide some intuition for our result.

The impossibility of herd behavior stems from the fact that neo-additive CEU traders' beliefs are anchored around α . As a result, their belief updating process exhibits a strong degree of sluggishness. CEU traders show particular reluctance in following the crowd. Consider for instance a low signal type trader. Since she sells initially by assumption, she can only engage in buy herding. When she observes a price increase, the additive part of the CEU belief, $E_{\pi_t}[V|S_0]$, increases in line with Bayes' rule as in Avery and Zemsky (1998). Since, however, she perceives ambiguity regarding the validity of her Bayesian asset valuation she tends to rely on her gut feeling to some extent, i.e. her individual degree of optimism. Under GBU, α is, however, unaffected by the price increase. As a consequence, the upward revision of $CEU[V | S_0, H]$ turns out to be smaller than the corresponding belief revision of her SEU counterpart from the model of Avery and Zemsky. Consequently, since $E_{\pi}[V|S_0] < bid(\pi) \leq bid^{CEU}(\pi)$ for all π , S_0 keeps selling a fortiori after having observed a price increase given that she sells to begin with.³³ This precludes her from ever engaging in buy herding behavior. The argument is symmetric for the high signal.

The mechanism preventing herding is, at the same time, the key driver for contrarianism. The sluggish belief updating makes neo-additive CEU traders prone to act against the crowd. As the asset price approaches one of the possible true states, i.e. if $\pi_t \rightarrow 1$ (or $\pi_t \rightarrow 0$), both informed trader types will eventually start selling (or buying) the asset, regardless of their initial trading decision. This is due to the fact, that their non-additive belief component

³²We can enforce this condition by setting $\alpha_S = E[V | S]$. The effects of dropping this assumption are discussed as we prove Theorem 4.3 in the Appendix.

³³To see that $E_{\pi}[V|S_0] < bid(\pi) \leq bid^{CEU}(\pi)$ holds, review Proposition 2.1 and Lemma 4.1.

bounds away their asset valuation from 1 and 0, as the public becomes increasingly confident about either state. Assume for instance a price decrease. As $\pi_t \rightarrow 0$, the quoted bid and ask prices also approach zero. At the same time, the non-additive part of the informed traders CEU beliefs $\delta_S \alpha_S$ is bounded away from 0.³⁴ Given their initial trading decisions the low signal can only engage in *buy* contrarianism while the high signal can only engage in *sell* contrarianism.

4.3 Investor Behavior in the CEU Model with Varying α

We now investigate investor behavior under the assumption that the individual degree of optimism α varies with the price π_t .

4.3.1 Irrational Exuberance and Herd Behavior

In this section we derive necessary and sufficient conditions for herd behavior in the CEU model with varying α . We begin our analysis by deriving a necessary condition.

Necessary Condition: The essential finding is that the degree of optimism α has to move pro-cyclically, i.e. increase with the market price. In addition, the individual reaction to market-wide optimism (pessimism) needs to be strong enough, i.e. informed CEU traders need to become particularly exuberant (desperate).

Theorem 4.4. Necessary Condition for Herding

Consider the CEU model with varying α .

*If **buy herding** occurs with positive probability, then $\exists \pi \in (\pi_0; 1) : \alpha_{S_0}(\pi) > ask(\pi)$.*

*If **sell herding** occurs with positive probability, then $\exists \pi \in (0; \pi_0) : \alpha_{S_1}(\pi) < bid(\pi)$.*

Proof: Due to symmetry reasons, we show only the buy herding statement. (A1) implies that S_0 sells initially while S_1 buys initially. Hence, only S_0 can buy herd (*BH1). If S_0 buy herds at some price $\pi > \pi_0$ (*BH3), then $CEU_{S_0}(\pi) > ask(\pi)$ (*BH2). Consequently,

$$(1 - \delta_{S_0})E_\pi[V \mid S_0] + \delta_{S_0}\alpha_{S_0}(\pi) > ask(\pi).$$

³⁴There are some peculiarities if $\alpha = 1$ or $\alpha = 0$. These cases of pure optimism and pessimism are discussed in the Appendix.

Solving this inequality for $\alpha_{S_0}(\pi)$ after having added and subtracted $\delta_{S_0}ask(\pi)$ on the r.h.s. of the inequality, yields

$$\alpha_{S_0}(\pi) > \frac{\delta_{S_0}ask(\pi) + (1 - \delta_{S_0})ask(\pi) - (1 - \delta_{S_0})E_\pi[V | S_0]}{\delta_{S_0}}.$$

Now noting that $ask(\pi) > E_\pi[V | S_0]$, we infer, that the r.h.s. of the inequality is greater than $ask(\pi)$, which proves the statement. □

In line with the market maker's price setting derived in Lemma 4.1, *bid* and *ask* in Theorem 4.4 refer to the bid and ask prices that are quoted in the similarly parameterized baseline herd model without ambiguity.

Let us provide some additional intuition regarding the buy herding condition. (A1) implies that only S_0 sells initially. In particular, the low signal initially values the asset at $E_{\pi_0}[V | S_0]$, i.e. she is neither particularly optimistic nor pessimistic. Even as the asset price π appreciates, the additive part of the low signal's asset valuation $E_\pi[V | S_0]$ remains well below the ask price, compare Section 2. Hence, S_0 will only decide to buy at some price $\pi_t > \pi$ if her degree of optimism $\alpha_{S_0}(\pi)$ is large enough to compensate for this fact. Since $\alpha_{S_0}(\pi_0) = E[V | S_0] < E_\pi[V | S_0]$, this means that it is necessary for buy herding, that $\alpha_{S_0}(\pi) \gg \alpha_{S_0}(\pi_0)$, i.e. S_0 's degree of optimism has to increase with the asset price. An incremental rise of α_{S_0} would, however, be insufficient.

The minimum requirement for S_0 to value the asset above the ask price is $\alpha_{S_0} > ask > \pi$. Noting that $\alpha_{S_0}(\pi_0) < \pi_0$, this implies that buy herding requires that $\alpha_{S_0}(\pi) - \alpha_{S_0}(\pi_0) \gg \pi - \pi_0$. Such a disproportionate surge in individual optimism compared to the increase of optimism exhibited by the market, can well be interpreted as (possibly unwarranted) exuberance on the part of the S_0 type trader. The intuition is similar for sell herding of S_1 . We would, however, label the required disproportionate increase in individual pessimism as (potentially exaggerated) desperation of the S_1 type traders.

Sufficient Condition: Before stating the formal sufficient condition for herding, let us develop some intuition first. Sufficiency for e.g. buy herding requires that at some point the degree of optimism α surmounts the ask price for good, i.e. $\exists \pi^* > \pi_0$, where $\pi^* < 1$ such

that $\alpha(\pi) > ask(\pi)$ for all prices $\pi \in (\pi^*; 1)$. If such an optimism function coincides with high primary ambiguity δ_0 , then this is sufficient for S_0 to buy the asset, at least for prices in a neighborhood of 1.

If δ_0 is large the trader's lack of confidence in her additive belief component is strong. Consequently, her asset valuation is strongly biased towards the non-additive component α . If her faith in her gut-feeling is sufficiently strong, $\alpha(\pi) > ask(\pi)$ over-compensates the fact that her additive belief component $E_\pi[V \mid S_0] < ask(\pi)$ driving her asset valuation $CEU_{S_0}(\pi)$ above the ask price for π close to 1. If primary ambiguity is too low, then no amount of optimism will ever drive the low signal's valuation of the asset above the ask price.³⁵

This sufficiency condition is in line with the intuition that for example retail traders are more prone to herding than professionals. Indeed, the worse a trader's understanding about financial markets in general and the functioning of a particular financial asset, the higher her degree of perceived primary ambiguity and the more likely, that she will eventually engage in herd behavior.

Theorem 4.5. *Sufficient Condition For Herding*

Consider the CEU model with varying α .

Let the level of primary ambiguity

$$\delta_0 > 1 - \frac{\mu(1 - q) + \theta}{q(\mu + 2\theta)},$$

where q denotes the signal precision, μ the informed trader share and $\theta := (1 - \mu)/3$ is the probability that a noise trader buys, sells or holds one unit of the asset.

*If $\alpha_{S_0}(1) = 1$, then **buy herding** occurs with positive probability.*

*If $\alpha_{S_1}(0) = 0$, then **sell herding** occurs with positive probability.*

Proof: The mathematical proof is again left to the Appendix. The principal idea of the proof has been outlined prior to the statement of Theorem 4.5.

□

A few remarks regarding Theorem 4.5 are in order.

First, the requirement that $\alpha_{S_0}(1) = 1$ ($\alpha_{S_1}(0) = 0$) has intuitive appeal. In the limiting

³⁵Compare the case of a pure optimist in the Appendix.

case, when the market becomes confident about either state of the world beyond any doubt, i.e. if risk vanishes completely, then even CEU traders who perceive ambiguity should value the asset at 1 and 0 respectively.

Second, we note that the minimum required amount of primary ambiguity δ_0 depends on the informed trader share μ and the signal precision q . Comparative static analyses reveal that $\delta_0^* = 1 - \frac{\mu(1-q)+\theta}{q(\mu+2\theta)}$ increases in both μ and q .³⁶ That is, an increase in μ and q tends to reduce investor proclivity to engage in herd behavior. This appears to be intuitive in the case of q . Better informed traders should *ceteris paribus* be less easily swayed by the crowd to change their trade decisions. In the case of μ the result is driven by the fact that the market maker faces a higher risk that his counter-party is informed. To compensate for that risk, he quotes a higher bid-ask spread which makes extreme switches of traders from selling to buying and vice versa less likely, compare the discussion of the impact of information risk on herding intensity in Paper ?? of this thesis.

Finally, we note that the way Theorem 4.5 is stated, it hinges on (A2). A general version of the sufficiency result that does not require (A2) is provided in Section 8.B in the Appendix of this paper.

The question remains, whether we expect to observe herding implied by Theorems 4.4 and 4.5 in the real world.

Since there is strong evidence that *risk* aversion moves pro-cyclically, see e.g. Bollerslev et al. (2011), we conjecture that ambiguity aversion or optimism should exhibit similar features given the conceptual proximity of these preference parameters.

Moreover, we would argue that there are abundantly many real-world examples where investor behavior showed corresponding characteristics. Popular precedents are the dot-com bubble at the turn of the millenium, the recent US house price bubble as well as the bubble of the Shanghai Composite Index in 2015.

Finally, the relevance of investor herding under ambiguity can be motivated micro-economically by the findings of Heath and Tversky (1991). They provide experimental evidence that once the judged probability of an ambiguous event is high, individuals tend to become ambiguity loving. If the price π_t is in a neighborhood of 1, even an S_0 type trader is fairly confident that the high state is true. Hence, according to Heath and Tversky (1991),

³⁶The formal derivation of these results is based on elementary calculus and has, thus, been omitted from the paper.

S_0 should assign a higher probability to the high state than prescribed by her additive belief component.³⁷ This would support the idea of Theorem 4.5 that (potentially irrational) exuberance and despair may drive investor herding at least for prices close to 1 and 0.

A Class of Optimism Functions Allowing for Investor Herding: We provide a class of optimism functions $\alpha(\pi|q, \pi_0)$ that are sufficient for herd behavior if $\delta_0 > 1 - \frac{\mu(1-q)+\theta}{q(\mu+2\theta)}$.

For the low signal, we have

$$\alpha_{S_0}(\pi_t) = \begin{cases} E[V | S_0, \pi_t] + E[V | S_0, \pi_t] \frac{\pi_t - \beta_1}{\beta_1}, & \text{if } \pi_t \leq \beta_1 \\ E[V | S_0, \pi_t] + (1 - E[V | S_0, \pi_t]) \frac{\pi_t - \beta_1}{1 - \beta_1}, & \text{if } \pi_t > \beta_1, \end{cases} \quad (5)$$

where $\beta_1 \geq \pi_0$. If $\beta_1 = \pi_0$, then in line with (A1) the corresponding CEU asset valuation of the low signal is initially equal to the additive component, i.e. $CEU_{S_0}(\pi_0) = E[V | S_0]$. The larger β_1 , the more pessimistic S_0 and the longer it takes before the low signal eventually becomes exuberant. S_0 needs to observe a much stronger buy side accumulation of traders before she will decide to follow the crowd and buy the asset. Indeed, note that for $\beta_1 > \pi_0$, we have $CEU_{S_0}(\pi_0) < E[V | S_0]$ which means that S_0 initially has a pessimistic view on the investment opportunity. She needs to overcome this a priori skepticism before she becomes inclined to invest in V . As long as $\beta_1 < 1$, sufficiency for buy herding as implied by Theorem 4.5 holds.

Similarly for the high signal, we have

$$\alpha_{S_1}(\pi_t) = \begin{cases} E[V | S_1, \pi_t] + E[V | S_1, \pi_t] \frac{\pi_t - \beta_2}{\beta_2}, & \text{if } \pi_t \leq \beta_2 \\ E[V | S_1, \pi_t] + (1 - E[V | S_1, \pi_t]) \frac{\pi_t - \beta_2}{1 - \beta_2}, & \text{if } \pi_t > \beta_2, \end{cases} \quad (6)$$

where $\beta_2 \leq \pi_0$. The interpretation of β_2 for the high signal is symmetric to the interpretation of β_1 for the low signal.

³⁷Note that strong pessimism also reflects ambiguity lovingness in the sense of Heath and Tversky (1991). The mere difference is that a high degree of confidence regarding the low state causes CEU traders to assign a higher probability to the low state, than their additive belief would dictate.

4.3.2 Contrarianism With Varying α

The intuition and mechanisms driving contrarianism are the same for varying α as under GBU. Thus, we do not provide formal necessary and sufficient conditions. We point out, however, that necessary conditions for contrarianism can be stated in a similar fashion as for herding. One simply has to exchange the intervals from which π is chosen in Theorem 4.4. For sufficiency, consider a CEU trader, whose degree of optimism α essentially stays constant for all $\pi \in [\epsilon; 1 - \epsilon]$ or even changes in a countercyclical fashion, then by similar arguments as in Section 4.2, the initial valuation of the low (high) signal would rise above (drop below) the ask (bid) price as the price decreases (increases).

Now, that we have derived fairly general conditions for herding and contrarianism in the CEU model, we shall shift our focus on the analysis of market outcomes and social learning.

4.4 Market Outcomes and Social Learning in the CEU Model

This section illustrates that herding and contrarianism in the CEU model have an equal potential to prevent the market from learning about the asset's fundamental value and to move prices away from fundamentals.

Monotonicity of the informed trader expectations, i.e. $CEU[V \mid S_0, H_t] < CEU[V \mid S_1, H_t]$, implies that whenever an informed trader type engages in herding or contrarianism all informed traders take the same action. That is, both types of investor behavior necessarily lead to an informational cascade. If an informational cascade occurs at t , social learning stops and the price fixes at π^* until the end of trading, i.e. $\pi_\tau \equiv \pi^*$ for all $t < \tau \leq T$. Such a price consensus is inherently inefficient since it prevents the market from learning about the asset's true value and consequently from pricing the asset at its fundamental, compare Chamley (2004).

In that sense, herding as well as contrarianism in the CEU model lead to inefficient market outcomes. This would be of minor concern if prices generally moved towards the asset's fundamental until the informational cascade takes place. That is, if $V = 1$ ($V = 0$), we observed $\pi^* > \pi_0$ ($\pi^* < \pi_0$) with high probability. This is, however, not necessarily the case as we illustrate by discussing Figures 3 and 4.

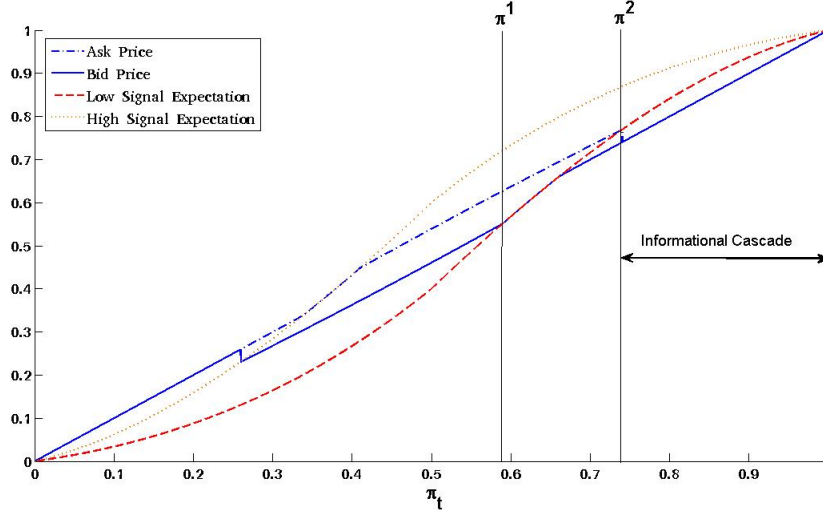


Figure 3: Irrational exuberance and buy herding

Notes: Informed trader asset valuations CEU_S , bid price bid^{CEU} and ask price ask^{CEU} are depicted with respect to the price π_t . The primary ambiguity is $\delta_0 = 0.5$, the informed trader share is $\mu = 0.3$, the initial prior is $\pi_0 = 0.5$ and the signal precision is $q = 0.6$. $\alpha(\pi_t)$ is given by Equations (5) and (6) with $\beta_1 = \beta_2 = 0.5$.

Figure 3 depicts a situation, where the low signal S_0 (the high signal S_1) engages in buy (sell) herding with positive probability. We focus only on the buy herding case. Initially, at $\pi_0 = 0.5$, the low signal values the asset as if she was an SEU maximizer, i.e. $CEU_{S_0}(\pi_0) = E_{\pi_0}[V | S_0]$. The low signal's optimism function α_{S_0} ensures that her asset valuation $CEU_{S_0}(\pi)$ is highly elastic with respect to the degree of optimism exhibited by the market. Indeed, as S_0 observes an increasingly strong price upsurge, she not only contracts the optimistic market sentiment but really becomes *overly* enthusiastic regarding the prospect of investing into the risky asset V .

As the price π rises above π^1 , the low signal changes her trading decision from selling to holding. In line with Lemma 4.1, the quoted bid price is equal to S_0 's valuation of the asset. As $CEU_{S_0}(\pi)$ becomes greater than π_t , the market maker quotes a bid price equal to π_t . If additional buys are observed, the price π_t eventually rises above π^2 . At that point, S_0 's exuberance causes her to start buy herding. An informational cascade occurs, since all informed trader types buy at that point. Social learning stops and the price is fixed at $\pi^* = 0.75$.³⁸

³⁸Note, that the discontinuity of the ask price at $\pi = 0.26$ indicates that the symmetrically modeled high signal starts selling. An informational cascade occurs, because at that point all informed traders sell and the

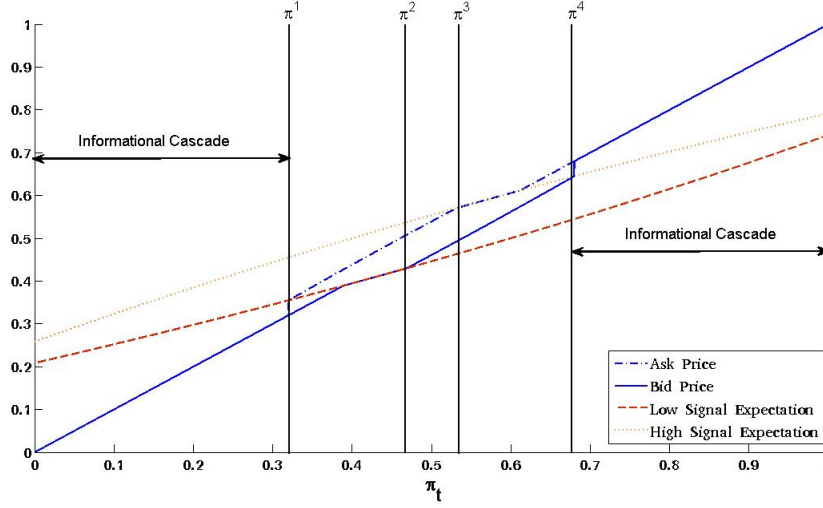


Figure 4: Market outcomes under contrarianism

Notes: Informed trader asset valuations CEU_S , bid price bid^{CEU} and ask price ask^{CEU} are depicted with respect to the price π_t . The degree of optimism is $\alpha = 0.5$ for both informed trader types, the primary ambiguity is $\delta_0 = 0.3$, the initial prior is $\pi_0 = 0.5$, the informed trader share is $\mu = 0.3$ and the signal precision $q = 0.6$.

To see that $\pi^* = 0.75$ becomes the price consensus with relevant probability even if the low state is true, i.e. if $V = 0$, note that S_0 engages in buy herding if the trade history contains at least 7 more buys than sells. Even though $V = 0$, there is ample potential for a buy side accumulation of traders. All high signals and one third of the noise traders buy the asset upon arrival. As long as there is no informational cascade, the buy probability, i.e. the probability of a price increase, is $\mu(1 - q) + \theta = 0.353$ for the model parameterization of Figure 3. Similarly, the probability to observe a price decrease is $\mu q + \theta = 0.413$. Given the high short-term dispersion of a price-process that is governed by these probabilities, it is apparent that the likelihood of prices moving away from fundamentals due to herding is far from negligible. The situation is symmetric for sell herding if $V = 1$.

Figure 4 illustrates the case where the low signal (high signal) may become a buy (sell) contrarianist. In line with the GBU rule, $\alpha = 0.5$ for both informed trader types. Assuming that $\pi_0 = 0.5$, S_0 sells initially and S_1 buys initially. Let us focus on the buy contrarianism case. As sells are observed, the asset price π decreases. As π falls below π^2 , S_0 stops selling price is fixed at 0.25.

and starts holding. If additional sells cause the price to drop below π^1 , S_0 switches from holding into buying, thus acting as a buy contrarianist. Since at that point both informed trader types buy the asset, an informational cascade occurs. Social learning stops and the asset price remains constant at $0.3 = \pi^* < \pi^1$ for all remaining trade periods.

Now, assume that $V = 1$ is the true state. The probability of a price decrease in t is $\mu(1 - q) + \theta = 0.353$ as long as there is no informational cascade.³⁹ This is due to the fact that the low signal reveals her private information through her action regardless of whether she sells or holds, compare price updating in Lemma 4.2. If S_0 sells (holds), prices decrease after an observed sell (hold).

If the trade history contains at least 4 more sells than buys, S_0 engages in buy contrarianism. The market “agrees” with considerably positive probability to depreciate the asset value to $0.3 = \pi^*$ even though $V = 1$ is the true state.

To make these insights more precise, we derive a formula for the probability of such wrong cascades due to herding as well contrarianism in the following section. We can use this formula to conduct comparative statics on the likelihood that herders and contrarians move prices away from fundamentals.

4.5 Investors Moving Prices away from Fundamentals - Deriving the Probability of a Wrong Cascade

We have illustrated that herding and contrarianism in the CEU model may lead to price distortions. The aim of this section is to quantify the probability of such an event and investigate how this probability is related to other model parameters.

To keep things tractable, let us assume without loss of generality that either herding or contrarianism are possible but not both. In addition, we assume symmetry of α_{S_0} and α_{S_1} in the sense that the minimum number of price increases (decreases) after which the informed traders start herding or acting as contrarians are the same for the high and low signals.⁴⁰ We denote this number as n^* .

Under these assumptions the probability of prices moving away from fundamentals co-

³⁹The probability of price increase is $\mu q + \theta = 0.413$.

⁴⁰This also includes the GBU case of fixed α . The result easily generalizes to the case where the symmetry assumption is dropped. A look at the proof of Lemma 4.6 in the Appendix will reveal why this is true.

incides with the probability of a buy herding (sell contrarianism) induced cascade if $V = 0$ and vice versa a sell herding (buy contrarianism) induced cascade given $V = 1$. We obtain

Lemma 4.6. *Probability Of A Wrong Informational Cascade*

Consider the CEU model, where either herding or contrarianism is possible. Let n^ be the minimum number of price increases (decreases) after which the low (high) signal starts herding or acting as a contrarian. Then, the probability of an informational cascade where all informed traders buy herd (act as sell contrarians), given that $V = 0$, is equal to the probability of an informational cascade where all informed traders sell herd (act as buy contrarians), given that $V = 1$. This probability is given by*

$$P_{n^*} = \frac{1}{\exp\left(-\ln\left(\frac{\mu(1-q)+\theta}{\mu q+\theta}\right)n^*\right) + 1}. \quad (7)$$

Proof: The proof is fairly technical and, thus, details are left to the Appendix. The idea of the proof is that the probabilities to observe a price increase, decrease or constant prices are the same as long as there is no informational cascade. This makes the problem of calculating P_{n^*} equivalent to a two-sided gambler's ruin problem with $2n^*$ possible states. We can specify the transition probabilities in each state. Then standard techniques of linear algebra yield the desired result. □

To get a broader perspective of how the probability of wrong cascades reacts to shifts in model parameters, let us analyze the comparative statics of P_{n^*} .

Lemma 4.7. *Comparative Statics Of Price Distortions*

In the CEU model, the probability of prices moving away from fundamentals P_{n^}*

- *Decreases with the informed trader share μ ;*
- *Decreases with the signal precision q ;*
- *Increases with the degree of primary ambiguity δ_0 ;*
- *Increases with the degree of individual optimism α if investors are prone to herding;*
- *Decreases with the degree of individual optimism α if investors are prone to contrarianism;*

Proof: Since the proof is fairly technical, it is left to the Appendix. □

The results of Lemma 4.7 are fairly intuitive. Since μ can be seen as the quantity of fundamentally relevant information in the market and q can be viewed as the quality of the same, it is straight forward that an increase of both should reduce the likelihood that prices move away from fundamentals. If there are more and better informed traders, then prices should more accurately reflect the asset's true value.

Likewise an increase in primary ambiguity δ_0 will make investors more prone to irrational exuberance (desperation) in the case of herding and will make the belief updating of potential contrarian traders even more sluggish. In both cases, higher ambiguity will cause CEU traders to rely more on their gut feeling which may advise them to take wrong actions. Hence the probability of prices moving away from fundamentals increases. The different results for a change in α stem from the fact that we see α as a function in π . We define an increase in α as a general increase of individual optimism elasticity with respect to a change in market sentiment π .⁴¹ Now, consider a CEU trader who is prone to contrarianism. If her individual optimism reacts more elastically to price changes, then her belief updating tends to be less sluggish, thus, reducing her contrarian tendencies. Similarly if the CEU trader is prone to herding, a higher α implies that her willingness to ignore her private signal and follow the crowd increases.

We have investigated the conditions under which herding and contrarianism are possible in the CEU model. Both lead to informational cascades and, thus, prevent the market from confidently learning about the true state and may cause price distortions. In fact, we find that price distortions are equally likely under contrarianism and under herding. Given the antithetical nature of the two types of behavior, however, we would have expected that herding and contrarianism result in different market outcomes. To carve out these differences we shall depart from the assumption that investor preferences are common knowledge and consider a perturbed version of the model. We will discuss the details of this approach and its insights in the next section.

⁴¹Formally, α^1 is said to be greater than α^2 if and only if $\alpha^1(\pi) < \alpha^2(\pi) \forall \pi \in (0; \pi_0)$ and $\alpha^1(\pi) > \alpha^2(\pi) \forall \pi \in (\pi_0; 1)$.

5 Price Dynamics under Herding and Contrarianism - the Perturbed CEU Model

This section provides insights regarding differences of stylized price dynamics under herding and contrarianism, respectively. In the CEU model, however, herding and contrarianism lead to informational cascades, i.e. constant prices. To circumvent this issue, we assume that market participants exhibit marginal uncertainty regarding investor preferences. The resulting setup is called the perturbed CEU model because it resembles in spirit the concept of a perturbed game.

5.1 The Perturbed CEU Model

All definitions and assumptions from Sections 2 and 3 hold. In particular, all informed traders are CEU maximizers. Yet, now we assume that market participants perceive ambiguity regarding informed traders' preferences. That is, they do not fully discount the possibility that informed traders are ambiguity neutral. More precisely, the market believes that informed traders have neo-additive CEU preferences with probability $1 - \epsilon$ and are expected value maximizers as in the baseline model with probability ϵ for some arbitrarily small $\epsilon > 0$.⁴²

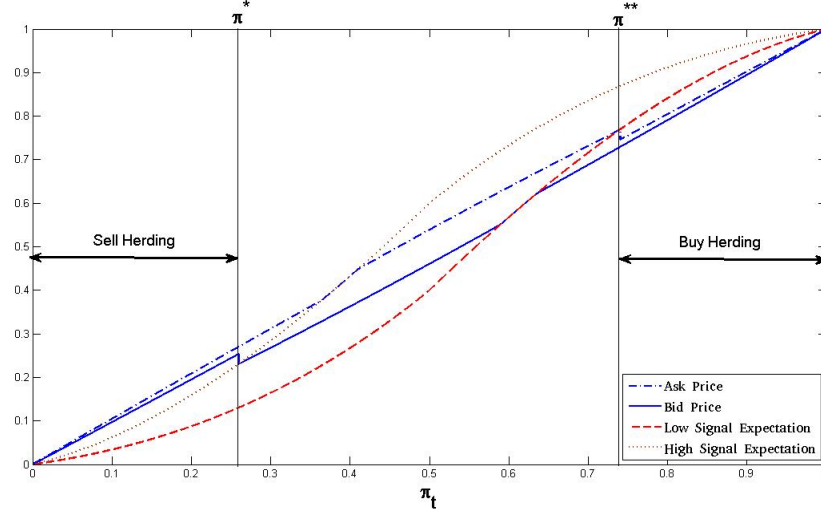
The most important property of the perturbed CEU model is the absence of (full) informational cascades. Figure 5 illustrates this fact by showing that even under herding and contrarianism the market maker quotes a positive bid-ask spread. This implies that the market still infers information from observed trade decisions a_t even if all informed traders take the same action in t . As a consequence, the price π_t continues to evolve under herding and contrarianism in the perturbed model.⁴³

The absence of informational cascades stems from the market's belief that a share of ϵ of the informed traders are expected value maximizers and, thus, behave as in the baseline model. Consider for instance a situation where both signals buy the asset (e.g. buy herding

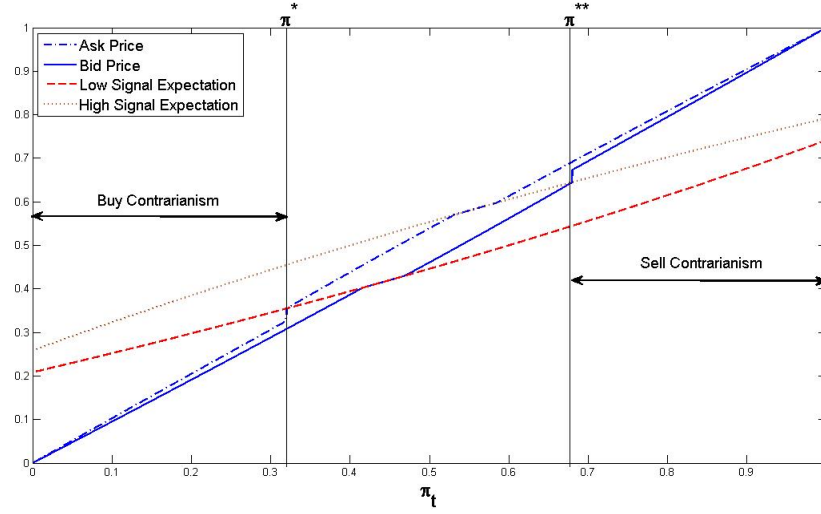
⁴²The fact that market participants perceive ambiguity regarding other informed traders' preferences is in line with the game theoretic literature, compare e.g. Eichberger and Kelsey (2014) and Eichberger and Kelsey (2000). In the context of these frameworks an agent's belief regarding the other players' ambiguity preferences may deviate from the truth.

We note that Ford et al. (2013) focus their analysis exclusively on the somewhat extreme case of $\epsilon = 1$.

⁴³The corresponding formal results on market maker price setting and price updating are notationally tedious in the perturbed CEU model and, thus, left to Section 8.C in the Appendix of this paper, see Propositions 8.6 and 8.7.



(a) Herding



(b) Contrarianism

Figure 5: Herding and contrarianism in the perturbed model

Notes: Figure (a) depicts informed trader asset valuations CEU_S , bid price $bid^{\epsilon-CEU}$ and ask price $ask^{\epsilon-CEU}$ with respect to the price π_t in a market prone to herding. For illustrative purposes, we set $\epsilon = 0.25$. The primary ambiguity is $\delta_0 = 0.5$, the informed trader share is $\mu = 0.3$, the initial prior is $\pi_0 = 0.5$ and the signal precision is $q = 0.6$. $\alpha(\pi_t)$ is given by Equations (5) and (6) with $\beta_1 = \beta_2 = 0.5$. Figure (b) depicts informed trader asset valuations CEU_S , bid price $bid^{\epsilon-CEU}$ and ask price $ask^{\epsilon-CEU}$ with respect to the price π_t prone to contrarianism. For illustrative purposes, we set $\epsilon = 0.25$. The primary ambiguity is $\delta_0 = 0.3$, the informed trader share is $\mu = 0.3$, the initial prior is $\pi_0 = 0.5$ and the signal precision is $q = 0.6$. $\alpha = 0.5$ for both informed trader types in line with GBU.

in Figure 5 (a)). The market believes that a share of ϵ of the S_0 type traders still acts as in the baseline model without ambiguity and sells the asset, compare Proposition 2.1. Hence, an observed sell is viewed to contain some information as the market does not fully discount the possibility that the trade is carried out by an S_0 type informed trader. Consequently, π_t decreases in line with Bayes' rule after a sell is observed. The argument is similar for a price increase after an observed buy.

Note, however, that the amount of information inferred from a trade under herding and contrarianism may be very small, particularly if ϵ is small. Consequently, herding and contrarian regimes can be seen as partial informational cascades in the sense of Avery and Zemsky (1998) and Park and Sabourian (2011).

The second key property of the perturbed model is that it inherits the results regarding the necessary and sufficient conditions for herding and contrarianism derived in the CEU model, see Theorems 4.3, 4.4 and 4.5. To see why this is true, assume that in line with (A1) the low signal sells initially while the high signal buys initially. As long as S_1 buys and S_0 does not, the market maker quotes the same ask price as in the CEU and baseline model without ambiguity.⁴⁴ Since S_0 's asset valuation is also not affected by the perturbation assumption, the conditions for CEU_{S_0} surpass the ask price, i.e. for S_0 to buy herd or act as a buy contrarian are the same as in the CEU model.

Indeed, since the model parameterizations in Figures 5 (a) and (b) coincide with the ones used for Figures 3 and 4, respectively (aside from the perturbation parameter ϵ), the cut-off prices for herding and contrarianism (π^*, π^{**}) from Figures 5 (a) and (b) are precisely equal to the corresponding cut-off prices depicted in Figures 3 and 4.

Since we have established that herding and contrarianism are possible in the perturbed CEU framework and that learning always continues we can now study how prices evolve under herding and contrarianism.

5.2 Prices under Herding and Contrarianism in the Perturbed CEU Model

Before deriving formal results we want to develop some intuition for the price dynamics by looking at the price process' transition probabilities under the different regimes.

⁴⁴Compare market maker pricing results for the CEU model (Lemma 4.1) and the perturbed model (Proposition 8.6 in Section 8.C in the Appendix).

	Probability of a Price Increase	Probability of a Price Decrease	Probability of a Constant Price
Buy Herding	$\mu + \theta$	θ	θ
Sell Contrarianism	θ	$\mu + \theta$	θ
Sell Herding	θ	$\mu + \theta$	θ
Buy Contrarianism	$\mu + \theta$	θ	θ

Table 1: Transition probabilities for π_t under herding and contrarianism

Notes: This table reports the transition probabilities of the price process in the perturbed CEU model given that CEU traders herd or act as contrarians respectively. μ is the share of informed traders. $\theta = (1 - \mu)/3$ is the probability for a noise trader to either buy, sell or hold.

Table 1 reports the probabilities of whether π_t moves up, down or remains constant given that investors herd or act as contrarians. We note that under all four regimes a price increase (decrease) coincides with an observed buy (sell). As a consequence, the probability for a price increase (decrease) is the same as the probability of a buy (sell). The same argument applies for constant prices and observed holds.⁴⁵

Consider for instance the case of buy herding. The total probability of observing a buy is the probability of an informed buy plus the probability of a noise buy. Since under buy herding all informed traders buy the asset, the probability of an informed buy is μ . Similarly, since one third of the noise traders buys the asset, the probability of observing a noise buy is $\theta = (1 - \mu)/3$. Thus, the total probability of observing a buy under buy herding and, hence, a price increase is $\mu + \theta$. Likewise, since the only traders selling under buy herding are noise traders, the probability of observing a sell and, thus, a price decrease is θ .

By a similar line of reasoning the probability of a price decrease under sell contrarianism is $\mu + \theta$. Since only noise traders buy under sell contrarianism, the probability of a price increase is θ .

Avery and Zemsky (1998) show that the price in their model eventually converges to 1 if the majority of the informed traders buys while it converges to 0 if the majority of the

⁴⁵Compare the price updating rules in the perturbed model summarized in Proposition 8.7 in Section 8.C in the Appendix for details.

informed traders sells.

The same principal should govern the price process in the perturbed CEU model under herding and contrarianism. We conjecture that buy herds in the perturbed CEU model push prices towards 1 (the majority buys), while sell contrarianists pull it back towards π_0 (the majority sells). Similarly, sell herds should push the price towards 0 while buy contrarians pull it back up towards π_0 .

These hypotheses are confirmed by the following two propositions.

Proposition 5.1. *Prices in the Perturbed Model - Herding*

In the perturbed CEU model, let $\pi^ < \pi_0$ ($\pi^{**} > \pi_0$) be the cut-off prices, such that S_1 sell herds for all $\pi < \pi^*$ (S_0 buy herds for all $\pi > \pi^{**}$).*

Then the market will become confident regarding the low (high) state with positive probability, regardless of the true state of V .

Proof: We discuss the proof for the buy herding case only. The sell herding case is symmetric. For the buy herding statement we need to show that $P(\lim_{t \rightarrow \infty} \pi_t = 1) > 0$. First note that by the law of total probability, we have

$$\begin{aligned} P(\lim_{t \rightarrow \infty} \pi_t = 1) \\ = P(\lim_{t \rightarrow \infty} \pi_t = 1 | \exists \tau \geq 0 : \pi_\tau > \pi^{**}) P(\exists \tau \geq 0 : \pi_\tau > \pi^{**}). \end{aligned} \tag{8}$$

The second probability on the r.h.s. of Equation (8) is greater zero by the assumption that buy herding is possible. For the first probability on the r.h.s. of Equation (8), we define $\pi_t^{BH} := (\pi_t | \pi_t > \pi^{**})$ to be the price process under buy herding. Observe that π_t^{BH} is a sub-martingale with respect to the history H_t , i.e. $E[\pi_{t+1}^{BH} | H_t] > \pi_t^{BH}$.⁴⁶ Then, the martingale convergence theorem implies that $\pi_t^{BH} \rightarrow \Pi$ for $t \rightarrow \infty$ almost surely, i.e. $P(\lim_{t \rightarrow \infty} \pi_t^{BH} = \Pi) = 1$.

Since, the sub-martingale property implies that $\pi_t^{BH} \in (\pi^{**}, 1)$ increases almost surely for $t \rightarrow \infty$ and since $\pi_t^{BH} < 1$ for all t by definition, it follows that $\Pi = 1$.

This implies that $P(\lim_{t \rightarrow \infty} \pi_t = 1 | \exists \tau \geq 0 : \pi_\tau > \pi^{**}) > 0$ and, thus, concludes the proof that a market prone to buy herding will become confident regarding the high state with positive probability regardless of the true state of V .

⁴⁶The proof for this is left to Section 8.C in the Appendix.

□

Proposition 5.2. Prices in the Perturbed Model - Contrarianism

In the perturbed CEU model, let $\pi^ < \pi_0$ ($\pi^{**} > \pi_0$) be the cut-off prices, such that S_0 acts as a buy contrarian for all $\pi < \pi^*$ (such that S_1 acts as a sell contrarian for all $\pi < \pi^*$). Then the price will rise above π^* (drop below π^{**}) again almost surely.*

Proof: Consider the sell contrarian case. We need to show that $P(\exists \tau > t : \pi_\tau \leq \pi^{**} | \pi_t > \pi^{**}) = 1$.

Similar to before, we define $\pi_t^{SC} := (\pi_t | \pi_t > \pi^{**})$. We observe that π_t^{SC} is a super-martingale with respect to the history H_t , i.e. $E[\pi_{t+1}^{SC} | H_t] < \pi_t^{SC}$.⁴⁷ In other words, prices fall almost surely as long as there is sell contrarianism. Moreover, there exists a $\tilde{\pi}_t > \pi^{**}$ such that $\tilde{\pi}_{t+1} < \pi^{**}$ if there is a sell in t . Both arguments together yield that the price will drop below π^{**} almost surely given there is sell contrarianism at some time t . The argument is symmetric for buy contrarianism.

□

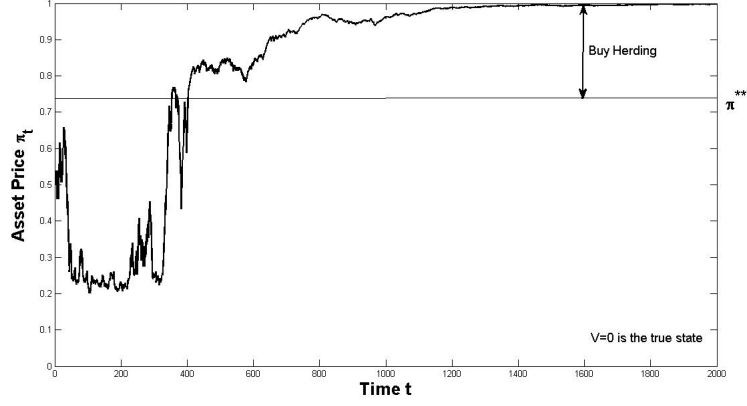
Propositions 5.1 and 5.2 state that the price evolves quite differently under herding and contrarianism in the perturbed CEU model. According to Proposition 5.1, herding causes the market to become confident about one of the states. A buy herd eventually drives the price towards 1 while a sell herd drives the price towards 0. Since this may happen regardless of the true state of V , the perturbed CEU model predicts that the market herds on the wrong state with positive probability.⁴⁸

In line with Proposition 5.2, contrarianism prevents the market from learning about the true state and anchors the price on some interval $\pi^* < \pi_0 < \pi^{**}$. Hence, both types of investor behavior still distort prices but they do so in very dissimilar ways.

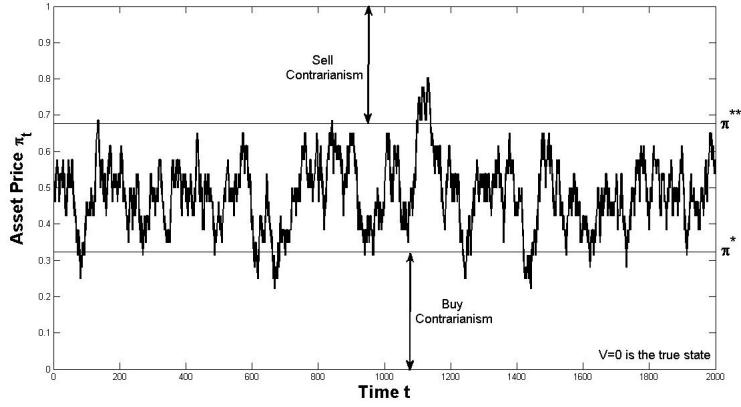
The different stylized price movement under herding and contrarianism in the perturbed CEU model are illustrated in Figure 6.

⁴⁷The proof for this is left to Section 8.C in the Appendix.

⁴⁸Note that this result is partly driven by the assumption that the market operates under the wrong assumption that a share of ϵ traders are SEU maximizers. Still, it provides qualitatively valuable insights as it is fair to assume that investor preferences are typically not accurately estimated in the real world. Since, moreover, there is a strong consensus regarding the validity of the efficient market hypothesis, we conjecture that markets tend to underestimate the share of investors deviating from SEU.



(a) Confident Herding On The Wrong State



(b) Anchored Price Movement und Contrarianism

Figure 6: Prices under herding and contrarianism

Notes: Figure (a) shows a simulated price path under herding for $T = 2000$ time periods. As before, the primary ambiguity is $\delta_0 = 0.5$, the informed trader share is $\mu = 0.3$, the initial prior is $\pi_0 = 0.5$ and the signal precision is $q = 0.6$. $\alpha(\pi_t)$ is given by Equations (5) and (6) with $\beta_1 = \beta_2 = 0.5$. The perturbation term is $\epsilon = 0.25$. Figure (b) shows a simulated price path under contrarianism for $T = 2000$ time periods. The primary ambiguity is $\delta_0 = 0.3$, the informed trader share is $\mu = 0.3$, the initial prior is $\pi_0 = 0.5$ and the signal precision is $q = 0.6$. $\alpha = 0.5$ for both informed trader types in line with GBU. The perturbation term is $\epsilon = 0.25$.

The herding case is depicted in Figure 6 (a). In line with the previously illustrated model outcome in Figure 5 (a), S_0 engages in buy herding as soon as the price surpasses π^{**} .⁴⁹ The first buy herd forms right before period 400. We observe, that this buy herd is broken shortly thereafter. This is due to a random arrival of noise traders selling the asset who push the price below π^{**} . When the second buy herd starts after period 400, however, it persists until period $T = 2000$. The sub-martingale property now governs the price process resulting in a long term price increase towards 1. Thus, the longer the herd persists, the less likely it is broken. In line with Park and Sabourian (2011) we refer to this as the self-enforcing nature of herding. By period 800, the price is well above 0.9. For it to fall below $\pi^{**} = 0.74$ again, many consecutive (noise trader) sells would have to be observed. The probability for such an event is already very small. Indeed, since we expect π_t to increase further on average (sub-martingale property), the probability of the herd being broken vanishes as $t \rightarrow T$. Note that the market becomes confident that the asset's true value is 1, while V in fact is 0, i.e. the market confidently herds on the wrong state.

Also observe that the price is considerably less volatile when S_0 type informed traders buy herd compared to when they do not. This is in line with the characteristics of a partial informational cascade. Since the market is certain that most informed traders have CEU preferences and knows that they engage in buy herding as $\pi > \pi^{**}$, the informational content the market infers from an observed trade drops significantly under buy herding. Since the price dynamics are mainly determined by the inferable information from an observed trade, prices become less volatile when S_0 buy herds, compare Avery and Zemsky (1998) and Park and Sabourian (2011).⁵⁰

The price evolution under contrarianism is depicted in 6 (b). In line with Figure 5 (b), S_0 engages in buy contrarianism (S_1 engages in sell contrarianism) as soon as the price surpasses π^{**} (falls below π^*). In line with Proposition 5.2, the price mainly stays on $(\pi^*; \pi^{**})$. The contrarian regimes are always very short due to the self-defeating nature of contrarianism, compare Park and Sabourian (2011). As soon as the price exceeds π^{**} , for instance, sell contrarians pull the price below π^{**} again causing the regime to end rather quickly. As a consequence, contrarians prevent the market from becoming confident regarding either state and, hence, from learning. In $T = 2000$, the asset price is still very far away from the asset's

⁴⁹Note, that the model parameterizations in Figure 6 are exactly as the corresponding model parameterizations in Figure 5.

⁵⁰In the case of an informational cascade the information content of a trade is 0 and, thus, the price remains constant, compare the CEU model.

fundamental value $V = 0$.

While for given model parameterization the price evolution is always similar in a market prone to contrarianism, the outcome could have been different under herding. A sell herd could have driven the price *towards* the asset's fundamental value of 0. As a consequence, we want to study the a priori probability of a wrong herd similar to the CEU model.

5.3 Price-Distorting Herding in the perturbed model

Quantifying the probability of price-distorting herds in the perturbed model is more complicated than in the CEU model. When informed traders change their trade decisions in the perturbed model, the probability and the extend of price increases and decreases varies. Yet, our ability to derive an analytical formula for the probability of wrong cascades in the CEU model in Lemma 4.6 hinges on the fact that price changes and their probabilities are fix as long as there is no cascade and trivial when there is an informational cascade. Hence, we are not able to provide a similarly appealing result for the perturbed model. We can, however, leverage Lemma 4.6 to infer upper and lower boundaries for the probability of wrong herds:

Lemma 5.3. *Probability of Wrong Herds in the Perturbed Model*

Consider the perturbed CEU model, where only herding is possible and assume symmetry between the signals as in Lemma 4.6. Let P_{WH}^ϵ denote the probability of a wrong herd in the perturbed CEU model. Let n^ be the minimum number of price increases (decreases) after which the low (high) signal starts herding. Let $k^* < n^*$ be the minimum number of observed price increases (decreases) after which the low (high) signal starts holding. Then $P_{WH}^\epsilon \in [P_{n^*}; P_{k^*}]$, where P is as in Equation (7).*

Proof: See Appendix. □

In the perturbed model, the probability of wrong buy herds tends to be even larger than in the CEU model. Driver for this result is the market's erring assumption that a share of ϵ of the informed traders are SEU maximizers. As long as preferences are common knowledge (CEU model), the price process is a martingale and cannot exhibit long term trends away from the asset's fundamental. In the perturbed model, on the other hand, such a wrong trend is possible.

Let $V = 0$ be the true state and assume for illustrative purposes that $\epsilon = 1$, that is, the market assumes that all informed traders are SEU maximizers although they have CEU preferences. Once S_0 switches into holding, the market does not accurately adjust the asset price downward anymore. Instead of depreciating the price after an observed hold (CEU model), the market depreciates the price after a sell. Yet, the probability to observe a sell is only θ (noise trader sell) since S_0 holds. For π_t to be a martingale, however, the probability of a downward revision would need to be $\mu + \theta$.

Hence, price decreases occur less often than they should. This results in an upward drift in the price process, i.e. a trend away from $V = 0$ (sub-martingale property). This effect is amplified if S_0 engages in buy herding.

These trends have the overall effect to drive prices away from fundamentals with greater probability in the perturbed model than in the CEU model.

We note, that the comparative statics of the lower boundary of P_{WH}^ϵ are readily provided by Lemma 4.7. For the comparative statics of the upper boundary P_{k*} , we find that the effect of an increase in μ is, indeed. An increase in q has weaker effects on P_{k*} than on P_{n*} .

5.4 The Burst Of A Bubble

Given the initial motivation of this paper, we would like to conclude this section by illustrating how the CEU model can be leveraged to explain the formation and subsequent burst of bubbles. As we have seen in the previous sections, the perturbed CEU model can explain the formation of a bubble. Yet, it cannot endogenously produce the burst of bubble before the asset's true value is revealed after the final period T .

To overcome this issue, we will allow ambiguity in the perturbed CEU model to be exogenously removed at some period $\tau < T$. Indeed, it is conceivable that unexpected events like the September 11 attacks on the United States in 2001 or Mario Draghi's "whatever it takes" speech in 2012 may cause jumps in primary ambiguity δ_0 or remove it altogether. The result is illustrated in Figure 7.

The model parameterization used to simulate the price path in Figure 7 allows for (wrong) herding. The asset's fundamental value is $V = 0$. Due to noise trading and trading of high signals, we observe an increase in the price in the first 70 periods. Shortly after $t = 70$, the low signal type traders engage in buy herding for the first time. The buy herd is broken a

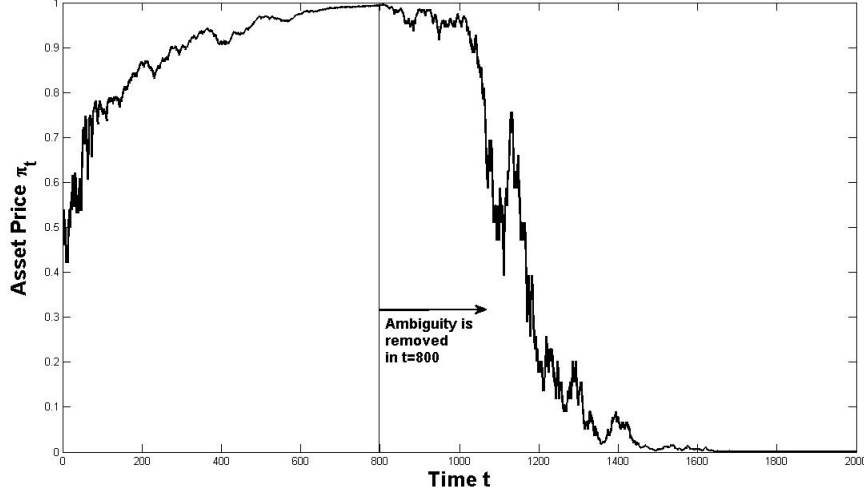


Figure 7: The formation and burst of a bubble

Notes: The figure shows a simulated price path for $T = 2000$ periods. The primary ambiguity $\delta_0 = 0.5$, the informed trader share is $\mu = 0.3$, the initial prior $\pi_0 = 0.5$ and the signal precision $q = 0.6$. $\alpha(\pi_t)$ is given by Equations (5) and (6) with $\beta_1 = \beta_2 = 0.5$. The perturbation parameter $\epsilon = 0.25$ for illustrative purposes. Primary ambiguity δ_0 is set equal to 0 after $t = 800$ periods. The true state is $V = 0$.

few times. A persistent buy herd forms only after period 110. Prices increase further towards 1 in line with Proposition 5.1. This continues until period 800, where some exogenous event removes the ambiguity, thereby bringing the herd to a halt.

Once ambiguity is removed, informed traders become expected value maximizers. Hence, S_0 type traders sell the asset. Since $V = 0$ is the true state, there are more S_0 type traders in the market than traders with a high signal. In line with the baseline model, we expect a price correction and that π_t eventually converges towards 0.

Since the informed traders' signal precision $q = 0.6$ is relatively low, it takes the market some time to learn that $V = 0$ is the true state. It is not before period 1000 that the market starts learning that the asset is wrongly priced. From period 1000 to 1300 the market corrects its assessment. We observe a sharp decline in the asset's value accompanied by regimes of increasing volatility. After period 1300 volatility diminishes as the market becomes increasingly confident regarding the low state and the standard learning mechanisms of Avery and Zemsky (1998) take predominant effect.

Event Study - The Subprime Mortgage Crisis: Let us apply these theoretical insights to real world events in the years 2002 to 2007. Consider Collateralized Debt Obligations (CDOs), highly complex and non-transparent credit derivatives that enjoyed unprecedented popularity among investors in the years leading up to the recent global financial crisis in 2007 and became essentially worthless once the underlying collaterals (sub-prime mortgages) defaulted in large numbers.

In a noteworthy statement of Warren Buffet in the annual report of Berkshire Hathaway in 2002, he outlaws complex derivatives such as CDOs as time bombs and financial weapons of mass destruction. He claims that these products depend on too many variables and have far too long times-to-maturity to be valued accurately. To make things worse, sub-prime mortgages, i.e. credits with a very high default risk, became a predominant collateral for CDOs between 2002 and 2007.

If we think of Warren Buffet's view as rational in an economic sense and take into account the high risk associated with the CDOs' collaterals, a rational assessment of CDOs should result in a low probability for the event "I earn money with CDOs (in the long run)". At the same time, Mr. Buffet's argument implies that the perceived ambiguity associated with CDOs should be high even among professional traders, i.e. δ_0 should be large. Indeed, since there was insufficient information regarding the actual default risks of the CDO's collaterals, let alone correlation structures of defaults, there was no way for investors to accurately assess the value of a CDO.

The incessantly high AAA-rating of CDOs by US rating agencies added to investors' perceived ambiguity as it contradicted their objective belief that CDOs with sub-prime mortgages as collaterals are inherently risky. In conjunction with increasing evidence of high returns and the general market sentiment - everybody was buying them - this contributed to a highly optimistic view of investors on the event "I earn money with CDOs (in the long run)". In neo-additive terms, this means that investors not only had a high δ but also a tremendously elastic α . These are the required ingredients for price distorting herding under ambiguity.

Only after default rates increased across subprime mortgages in the end of 2006 and the beginning of 2007, investors realized that they had erred. To speak in model terms, the ambiguity was removed - it then was clear that many CDOs were essentially worthless. Interestingly, this did not affect the CDO prices at first. Only as time passed and default rates skyrocketed, the market depreciated CDO prices to reflect their true value.

6 Robustness Discussion

6.1 Multiple Prior Setup With Smooth Ambiguity Functions

In this section we want address the question, whether our results hold if we choose a different approach to model ambiguity. For this, we translate the CEU model into a multiple prior setup and investigate whether similar results hold under the assumption that informed investors form beliefs according to smooth ambiguity functions as proposed by Klibanoff et al. (2005).

To model informational ambiguity in a multiple prior context, we follow the ideas provided by Gollier (2011) and assume that investors find it plausible that the precision of their signal is either q with probability 1 or it is uniformly distributed across $[0; 1]$. In addition, we assume that either distribution for q is considered to be equally likely, i.e. occurs with probability $d = 0.5$. This captures informational ambiguity in the sense that the informed trader has a lack of confidence that her private information signal is fundamentally driven. Thus, the informed trader's second-order belief d is closely related the degree of perceived ambiguity in the neo-additive CEU model. As a suitable family of smooth ambiguity functions, consider

$$\Phi(x) = \begin{cases} \exp(-ax), & \text{if } a \neq 0 \\ x & \text{if } a = 0 \end{cases}.$$

The parameter a would reflect the informed trader's degree of optimism similar to α in the CEU setup. If $a > 0$, the informed trader is pessimistic, for $a < 0$, she is optimistic. For $a = 0$, she is ambiguity neutral. The informed traders' asset valuation is given by

$$V_S = -\frac{1}{a} \log \left(d \int_0^1 \exp(-aE_q[V \mid S, \pi]) dq + (1-d) \exp(-aE_q[V \mid S, \pi]) \right) \quad (9)$$

if $a \neq 0$. While we can obtain results similar to Theorems 4.3, 4.4 and 4.5 as well as Lemmas 4.6 and 4.7 by the same arguments as in the neo-additive CEU model, we would loose some of the intuition provided in this paper. The parameters d and a cannot be linked as nicely to the parameters in the Avery and Zemsky (1998) framework, as δ_0 and α .

To gain some intuition how a pro-cyclical time varying a in the multiple prior setup can drive (potentially price-distorting) herd behavior, we suggest to set $a(p) = -K \tan(\pi(p - 0.5))$

for some $K > 0$, where in this case p represents the price and π the actual number π .

6.2 Risk Preferences

By the same line of reasoning as for the multiple prior setup with smooth ambiguity functions, we argue that the results of this paper can be reproduced in a framework where informed traders have varying risk preferences. If risk-preferences move pro-cyclically (counter-cyclically) in the same way ambiguity preferences do, we find that herding (contrarianism) becomes possible in the Avery and Zemsky (1998) framework.

To see this for herding, take Φ and $a(p)$ as before and note that the informed traders asset valuation becomes $V_S = \Phi^{-1}(E[\Phi(V) | S, \cdot])$. For contrarianism, repeat the exercise with fixed a .⁵¹

This is appealing from a theoretical perspective, in the sense that it generalizes the results presented in this paper. Indeed, experimental evidence from Drehmann et al. (2005) and Cipriani and Guarino (2005) suggests that investors in the Avery and Zemsky (1998) baseline model, do engage in contrarian behavior to some extent. This indicates that latent (or even counter-cyclical) risk preferences may play a role in decision-making under risk in social learning settings. The same experiments, however, generally find no evidence of herd behavior casting strong doubt on the economic relevance of pro-cyclical changes in risk preference for risky investment decisions.

We would argue that this does not impede the validity of the herding results derived in this paper. The choices in Drehmann et al. (2005) and Cipriani and Guarino (2005) are not choices under ambiguity. In line with Heath and Tversky (1991), we conjecture that investors facing ambiguity, exhibit ambiguity lovingness but not risk lovingness as they become confident about either state.

When viewing the ambiguity preference parameter α as a measure for optimism and pessimism its link to investor herding is also intuitively more appealing than for risk lovingness. While it is conceivable that investors contract optimism exhibited by the market and become overly enthusiastic regarding an investment opportunity, it is difficult to argue, why they would all of a sudden like the risk associated with a particular investment.

⁵¹This does not come as a surprise, since Décamps and Lovo (2006) show that differences in risk preferences between informed traders and market maker are sufficient for herding in a setup closely related to the Avery and Zemsky (1998) framework.

6.3 More States, More Different Types Of Traders

The results derived in this paper carry over to the more complex versions of Avery and Zemsky (1998) or the frameworks presented by Park and Sabourian (2011) and Cipriani and Guarino (2014). The important thing to note is that traders that perceive informational ambiguity may become prone to herding (contrarianism), even if they receive monotone private signals in the sense of Park and Sabourian (2011).⁵² Hence, herd (contrarian) behavior is not limited anymore to certain signal types like the u-shaped (hill-shaped) trader in Park and Sabourian (2011) or the low precision signal in Cipriani and Guarino (2014). As a consequence, herds or contrarianists in those models may consist of much larger shares of informed traders and, thereby, have the potential to drive prices away from fundamentals even in the long run.

6.4 Heterogeneous Attitudes Towards Ambiguity and Degrees Of Perceived Ambiguity

Instead of assuming a particular α for the informed traders, one could assume that the degree of optimism is distributed randomly across the population of informed traders. As the market grows more or less confident about either state, one could shift the mean and the variance of the distribution. If the mean of that distribution was to shift in a way that it fulfills the sufficient conditions for herding of Theorem 4.5 and its variance was connected to e.g. the variance of the public belief π_t , then herding would still be possible. In that case not all informed traders with a particular private signal would change their trading decision simultaneously and there would always be a certain amount of traders that do not change their decision to follow or act against the crowd. If the share of traders engaging in herd or contrarian behavior is large enough, however, price distortions are still possible in principle. Yet, if the distribution of the informed traders is part of the common knowledge structure, the price process will remain a martingale even under herding or contrarianism, compare e.g. Avery and Zemsky (1998). Hence, herds, while potentially persistently price-distorting, will not drive the price towards the wrong state in the long run. A detailed analysis of such a model would be an interesting avenue for future research.

Similarly, we could consider a heterogeneous population of informed traders that perceive different degrees of primary ambiguity. If the distribution of the primary ambiguity levels is part of the common knowledge structure of the model, the same argument applies

⁵²In that sense, every signal in a two-state world is monotone.

as before for the likely model outcomes and, thus should be included in future analyses.

7 Conclusion

We have provided a comprehensive framework to explain how ambiguity may affect investor behavior and social learning in the two-state, two-trader version of the Avery and Zemsky (1998) model. As we review Avery and Zemsky’s model setup without Knightian uncertainty, we observe that it implies that neither herding nor contrarianism are possible. We find that ambiguity and an invariant degree of optimism result in strong contrarian tendencies among informed traders but still preclude herd behavior. When allowing the individual degree of optimism to vary with market prices, herding becomes possible. It is necessary for herding, that the individual degree of optimism increases with market prices, i.e. with the degree of optimism exhibited by the market as a whole. If informed traders become overly exuberant or gloomy regarding an investment prospect they perceive a sufficiently high amount of primary ambiguity, then this is sufficient for herding to occur with positive probability.

We find, that herding and contrarianism alike cause informational cascades, thereby, preventing the market from pricing the asset at its fundamental value. Indeed, contrarianism as well herding moves prices away from fundamentals with positive probability. Such wrong cascades are qualitatively different for herding and contrarianism. If the market is marginally uncertain regarding investor preferences, informational cascades are only partial. In that case, the self-defeating nature of contrarianism bounds prices away from the asset’s potential fundamental values in the long run. The self-enforcing nature of herding, on the other hand, has the potential to drive price towards the wrong state.

Aside from the already mentioned theoretical extensions, the natural next step is to put the theoretical predictions of the CEU model to the test. Conducting experiments in a similar spirit as Drehmann et al. (2005) and Cipriani and Guarino (2005), where we add ambiguous components to the information signals to reflect the assumptions of the CEU model presented in this paper, would be an exciting avenue for future research. Similarly, one could test experimentally whether ambiguity regarding the distribution of the risky asset as proposed by Dong et al. (2010) leads to herding or contrarianism respectively.

From an empirical perspective, it may be interesting to investigate possibilities to measure the degree of primary ambiguity. If we assume for instance, that the absolute number of professional or expert traders for a particular asset is constant, then an increase in the number of traders would hint at an increase of average primary ambiguity. Similarly, one could investigate investor behavior in the aftermath of unexpected events, that are relevant

to the valuation of the asset. It would be interesting to analyze whether joint evidence of higher primary ambiguity and investor coordination have predictive power regarding future price reversals, that would be consistent with price-distorting herding.

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8 Appendix

This Appendix is structured as follows. Section 8.A provides an intuitive example why it is reasonable to assume that an economically relevant share of market participants perceives ambiguity when facing investment decisions. Section 8.B contains the all proofs omitted from the main part. Section 8.C provides the formal results for the perturbed model discussed in Section 5 in the main part. Section 8.D discusses market outcomes of a purely optimistic market under GBU, thus detailing the results of Section 4.2 in the main part. Section 8.E collects results from Avery and Zemsky (1998) that are relevant to this paper. Section 8.F discusses in further detail the relationship between different definitions for informational cascades and why we chose the one provided by Avery and Zemsky (1998). Section 8.G is a repository of the most important definitions and results from Chateauneuf et al. (2007) that are needed to confirm that our application of NEO-additive CEU preferences to the model of Avery and Zemsky (1998) is correct. Finally, Section 8.H discusses some inconsistencies of the GBU rule as prices approach 0 or 1, thereby, supporting the idea that it is reasonable that α varies with the price.

8.A Example of an Investor Facing Informational Ambiguity

As an example for informational ambiguity, consider a risk-neutral rational retail investor who has to decide whether or not to buy a particular stock (e.g. BMW - a German car manufacturer). She will make money on the investment if the price of the stock goes up, she will lose money if the price goes down.

She receives a recommendation from her online broker to buy the BMW stock but she has little knowledge about the German automotive industry (and BMW in particular). She knows her online broker is right 60 % of the times but she is not fully sure about her broker's agenda in this case because she sees that the BMW stock price has depreciated during the past month.

Given that she is rational and risk-neutral, she should buy the stock if and only if she believes that the price of the BMW stock will rebound with a probability greater than 50%. In the present case, however, the information does not enable her to determine the relevant probability exactly.⁵³

⁵³One could argue that she would only need to update an uninformative prior with a noisy signal according to Bayes' rule to conclude that the price will go up with 60% probability. Yet, this would only be true if the retail trader was confident that the online broker's success probability of 60% does apply to the current

There are many different ways the retail trader might process the online broker's recommendation and the stock price information to arrive at a probability assessment. She could think that her broker would not make such a recommendation if he did not have some valid information that the BMW stock price will go up. If such recommendations were made arbitrarily, the broker would risk to alienate and eventually lose customers who make losses as they follow the broker's recommendation. Despite BMW's recent downturn, she assigns a probability greater than 50% that the BMW stock price will go up again. Put differently, she retains an optimistic attitude towards the investment prospect. On the other hand, she might also think that the broker recommends buying the BMW stock because he wants to sell out his own BMW shares before the price drops further. This pessimistic view would lead to a probability assignment of less than 50% to an increasing stock price. Finally, she might retain a skeptical view on the online broker's recommendation but she might be less pessimistic than in the previous scenario. Since she knows little about cars and nothing about the broker's motivation for the recommendation, she might conclude that she could as well toss a fair coin to decide what to do, i.e. she assigns a probability of precisely 50% to an increasing stock price.

8.B Mathematical Proofs

In this section provide the remaining proofs of the main part of this paper.

Proof of Theorem 4.3: To prove this, we restate some of the results of Ford et al. (2013):

Lemma 8.1. *In the CEU model with $\delta_0 > 0$ and $\alpha \in [0; 1]$ fixed and $\pi_t \in [0; 1]$, we have*

- *The ask price in the Avery and Zemsky model is increasing and concave in π_t .*
- *The bid price in the Avery and Zemsky model is increasing and convex in π_t .*
- *$CEU_{S_1}(\pi_t)$ is increasing and concave in π_t .*
- *$CEU_{S_0}(\pi_t)$ is increasing and convex in π_t .*

All properties hold in a strict sense.

situation.

Proof: The proof is provided in Ford et al. (2013). □

With this, we can immediately prove the impossibility of herding in the CEU model with fixed α . We will conduct the proof for the impossibility of herding for the low signal S_0 . The proof is symmetric for S_1 .

Let us start with the impossibility of buy herding. For buy herding to be possible, S_0 has to sell initially, i.e.

$$x_1 := CEU_{S_0}(\pi_0) < bid^{CEU}(\pi_0) < \pi_0.$$

Moreover, we have that

$$y_1 := CEU_{S_0}(1) = (1 - \delta_{S_0}(1)) + \delta_{S_0}(1)\alpha = 1 - \delta_{S_0}(1)(1 - \alpha) \leq 1.$$

Now assume that the market price has increased, i.e. $\pi_t > \pi_0$. Then π_t can be written as a convex combination of π_0 and 1. That is, $\exists \lambda \in (0; 1)$ such that $\pi_t = \lambda\pi_0 + (1 - \lambda)$. In line with Lemma 8.1, convexity of $CEU_{S_0}(\pi_t)$ implies that $\forall \lambda \in [0; 1]$, we have

$$\lambda x_1 + (1 - \lambda)y_1 > CEU_{S_0}(\pi_t).$$

Since $x_1 > \pi_0$ and $y_1 \leq 1$, it immediately follows that

$$\lambda x_1 + (1 - \lambda)y_1 < \lambda\pi_0 + (1 - \lambda) = \pi_t \leq ask^{CEU}(\pi_t),$$

which in turn implies that $CEU_{S_0}(\pi_t) < ask^{CEU}(\pi_t) \forall \pi_t \in [\pi_0; 1]$. This is equivalent to the fact that S_0 never buy herds.

The assumption of monotonicity of the CEU asset valuation implies that if S_0 buys so does S_1 . This constitutes an informational cascade and, thus, S_0 can never sell herd.

For argument's sake, however, let us drop the monotonicity assumption for a moment. Sell herding would then still be impossible: Assume that S_0 buys initially (and S_1 does not). Hence, we have

$$x_1 := CEU_{S_0}(\pi_0) > \pi_0.$$

Moreover, we have again

$$y_1 := CEU_{S_0}(1) \leq 1.$$

Let $m_1 := \frac{y_1 - x_1}{1 - \pi_0}$. Then m_1 denotes the average slope of $CEU_{S_0}(\pi_t)$ on $[\pi_0; 1]$. Since $x_1 < \pi_0$ and $y_1 \leq 1$, it follows that $m_1 < 1$. Hence, since $CEU_{S_0}(\pi_t)$ is convex and increasing, it follows that

$$\frac{\partial CEU_{S_0}}{\partial \pi_t}(\pi_0) \leq m_1 < 1.$$

Again invoking that $x_1 > \pi_0$, this implies that the tangent Θ_1 of CEU_{S_0} in π_0 lies above π_t on $[0; \pi_0]$. Moreover, convexity of CEU_{S_0} implies that $CEU_{S_0} \geq \Theta_1$ for all π_t . Hence, we conclude that $bid^{CEU}(\pi_t) < \pi_t < \theta_1 \leq CEU_{S_0}$ for all $\pi_t \in [0; \pi_0]$, which precludes the possibility that S_0 engages in sell herding.

We continue the proof by showing the possibility of contrarian behavior for S_0 . Again, the argument is symmetric for the high signal.

Let S_0 sell initially, then only buy contrarianism is possible. Hence, we have to find a $\pi_t \in (0; \pi_0)$, such that $CEU_{S_0}(\pi_t) > ask^{CEU}(\pi_t)$. Noting that

$$CEU_{S_0}(0) = \delta_{S_0}(0)\alpha > 0 = ask^{CEU}(0),$$

continuity of CEU_{S_0} and ask^{CEU} implies that $CEU_{S_0} > ask^{CEU}$ in a whole neighborhood of zero, i.e. $\exists \epsilon > 0 : CEU_{S_0}(\pi_t) > ask(\pi_t) \forall \pi_t \in [0; \epsilon]$. But this already implies that S_0 engages in buy contrarianism $\forall \pi_t \in [0; \epsilon]$.

As before, monotonicity of CEU beliefs actually prevents sell contrarianism of S_0 from being possible. For argument's sake, let us drop this assumption for a moment. Then sell contrarianism of S_0 is indeed possible.

Let S_0 buy initially. For S_0 to act as a sell contrarian, we have to find a $\pi_t \in (\pi_0; 1)$, such that $CEU_{S_0}(\pi_t) < bid^{CEU}(\pi_t)$. Noting that

$$CEU_{S_0}(1) = (1 - \delta_{S_0}(1)) + \delta_{S_0}(1)\alpha = 1 - \delta_{S_0}(1)(1 - \alpha) < 1 = bid^{CEU}(1)$$

and invoking a continuity argument as before implies that there exists $\tilde{\epsilon} > 0$ such that $CEU_{S_0}(\pi_t) < bid(\pi_t) \forall \pi_t \in [1 - \tilde{\epsilon}; 1]$. Hence, S_0 may engage in sell contrarianism.

□

Note that for the impossibility of buy herding, we could have actually shown that $CEU_{S_0} < bid$ for all $\pi \in [\pi_0; 1]$. Intuitively, the argument would be that the increase of $CEU_{S_0}(\pi)$ in π is lower than the increase of the low signal's SEU belief $E[V \mid S_0, \pi]$ from the baseline model. Monotonicity in conjunction with convexity of CEU_{S_0} and $E[V \mid S_0, \pi]$ as well as the fact that $E[V \mid S_0, 1] = 1$ imply that CEU_{S_0} must remain below the bid price. That is, even weak forms of herding, i.e. switches from selling (buying) to holding are impossible if α is fixed.

Proof of Theorem 4.5: Again due to symmetry, we show only the buy herding statement. First, we note that $\alpha_{S_0}(1) = 1$, implies that $CEU_{S_0}(1) = 1 = ask(1)$, where $ask(\cdot) \geq ask^{CEU}(\cdot)$ denotes the ask price from the baseline model, see Equation (35).

Noting that (A2) implies that CEU_{S_0} is regular, we get

$$\begin{aligned} \frac{\partial CEU_{S_0}}{\partial \pi} &= \frac{\partial}{\partial \pi} [(1 - \delta_{S_0}(\pi))E_\pi[V \mid S_0] + \delta_{S_0}(\pi)\alpha_{S_0}(\pi)] \\ &= (1 - \delta_{S_0}(\pi))\frac{\partial}{\partial \pi}E_\pi[V \mid S_0] + \delta'_{S_0}(\pi)(\alpha_{S_0}(\pi) - E_\pi[V \mid S_0]) \\ &\quad + \delta_{S_0}(\pi)\alpha'_{S_0}(\pi) \end{aligned}$$

and evaluating it at $\pi = 1$, we get

$$\frac{\partial CEU_{S_0}}{\partial \pi}(1) = (1 - \delta_{S_0}(1))\frac{\partial}{\partial \pi}E_1[V \mid S_0] + \delta_{S_0}(1)\alpha'_{S_0}(1), \quad (10)$$

where we used the fact that $\alpha_{S_0}(1) = E_1[V \mid S_0] = 1$. Since $\alpha'_{S_0}(1) = 0$ due to (A2), we infer that CEU_{S_0} must be strictly increasing in 1 and thus also in a neighborhood of 1.

Moreover, $ask(\pi)$ is also strictly increasing in 1 and in a neighborhood of 1 (compare Lemma 8.1). For CEU_{S_0} to be greater than ask in a neighborhood of 1 it is thus sufficient, if

$$\frac{\partial CEU_{S_0}}{\partial \pi}(1) < \frac{\partial ask}{\partial \pi}(1). \quad (11)$$

Plugging in the right hand side of Equation (10) into the left hand side of Inequality (11) and using that $\alpha'_{S_0}(1) = 0$, we find that Inequality (11) is equivalent to

$$(1 - \delta_{S_0}(1))\frac{\partial}{\partial \pi}E_1[V \mid S_0] < \frac{\partial ask}{\partial \pi}(1). \quad (12)$$

Solving for $\delta_{S_0}(1)$ yields

$$\delta_{S_0}(1) > \frac{\frac{\partial}{\partial \pi} E_1[V | S_0] - \frac{\partial ask}{\partial \pi}(1)}{\frac{\partial}{\partial \pi} E_1[V | S_0]}. \quad (13)$$

Now observing that

$$\begin{aligned} \delta_{S_0}(1) &= \frac{\delta_0}{(1 - \delta_0)(1 - q) + \delta_0}, \\ \frac{\partial}{\partial \pi} E_1[V | S_0] &= \frac{q}{1 - q}, \\ \frac{\partial ask}{\partial \pi}(1) &= \frac{\mu(1 - q) + \theta}{\mu q + \theta}, \end{aligned}$$

we can solve Inequality (13) for δ_0 and obtain

$$\delta_0 > 1 - \frac{\mu(1 - q) + \theta}{q(\mu + 2\theta)},$$

which according to our initial argument is sufficient for the low signal to buy at prices in a neighborhood of 1. □

We note that if we were to drop assumption (A1), then Theorem 4.5 would hold trivially if the low (high) signal were to buy (sell) initially given that there is no informational cascade. For this, we would only have to exchange α_{S_0} and α_{S_1} in the buy and sell herding conditions. Then, the sufficient condition for sell herding implies that $\alpha_{S_0}(\pi) < bid(\pi) \leq bid^{CEU}$ after a sufficiently strong price drop and, thus, $CEU_{S_0}(\pi) < bid^{CEU}(\pi)$, which implies sell herding on the part of the low signal. The argument for S_1 is symmetric. A similar argument can be made if the definition for herding included switches from hold to buy and sell and the informed traders held initially.

If the second part (A2) is dropped (i.e. regularity still holds), then we require an additional condition regarding $\alpha'_{S_0}(1)$ and $\alpha'_{S_1}(0)$.

Corollary 8.2. General Sufficient Condition For Herding

Consider the CEU model with varying α , where (A1) and (A3) hold and α is sufficiently regular in π .

Let

$$C := \frac{K_1}{K_2} + \frac{(1 - \delta_0)}{\delta_0 K_2} (\mu + \theta)(1 - 2q),$$

where q denotes the signal precision, μ the informed trader share and $\theta := (1 - \mu)/3$ and $K_1 := \mu(1 - q) + \theta$ and $K_2 := \mu q + \theta$.

If $\alpha_{S_0}(1) = 1$ and $\alpha'_{S_0}(1) < C$, then **buy herding** occurs with positive probability.

If $\alpha_{S_1}(0) = 0$ and $\alpha'_{S_1}(0) < C$, then **sell herding** occurs with positive probability.

Proof: For the buy herding case, simply note that the $\alpha'_{S_0}(1)$ term does not disappear in Inequality (12). Then, solving it for $\alpha'_{S_0}(1)$ yields the condition $\alpha'_{S_0}(1) < C$. The argument for sell herding is identical. □

The boundary C implies that high primary ambiguity δ_0 still contributes towards the possibility of herding. The higher δ_0 , the less negative the second summand of C , the larger C and the less binding the slope condition for the optimism function. If, however, α increases too strongly in 1, i.e. $\alpha'_{S_0}(1) \geq K_1/K_2$, that is, the degree of optimism is rather inelastic with respect to changes in market sentiment for moderate prices, then no amount of primary ambiguity will lead to herd behavior.

In that sense, herding in the CEU model requires a departure from the certainty effect implied by prospect theory, compare e.g. Barberis and Thaler (2003). The certainty effect implies that individuals facing risk tend to undervalue probabilities close to 1 and overvalue probabilities close to 0. We would again appeal to the finding of Heath and Tversky (1991) that this is not necessarily the case for probability judgments under ambiguity.

Proof of Lemma 4.6: We will focus the proof on the herding case. The arguments are identical for the contrarian case.

Before we start the actual proof, let us state a supporting Lemma that will also help with our subsequent comparative static analysis.

Lemma 8.3. *Consider the CEU model. Let herding be possible and let wlog $\exists! \pi^* > \pi_0, \pi^{**} < \pi_0$ such that S_0 engages in buy herding for $\pi > \pi^*$ and S_1 engages in sell herding for $\pi < \pi^{**}$.*⁵⁴

⁵⁴This assumption is made for convenience. The sufficient condition only implies that there exist such π but not that they are unique. We would then have to consider the respective minimum or maximum over all such π .

Then Equations

$$\pi^* = \frac{(\mu q + \theta)^x \pi_0}{(\mu q + \theta)^x \pi_0 + (\mu(1 - q) + \theta)^x (1 - \pi_0)} \quad (14)$$

and

$$\pi^{**} = \frac{(\mu(1 - q) + \theta)^x \pi_0}{(\mu(1 - q) + \theta)^x \pi_0 + (\mu q + \theta)^x (1 - \pi_0)} \quad (15)$$

have unique solutions $x^* > 0$ and $x^{**} > 0$ respectively. Then $n^* := \lfloor x^* \rfloor + 1$ defines the minimum number of price increases the low signal has to observe before engaging in buy herding. Similarly, $n^{**} := \lfloor x^{**} \rfloor + 1$ defines the minimum number of price decreases the low signal has to observe before engaging in sell herding.

Proof: Uniqueness and positivity of the solutions of Equations (14) and (15) with respect to x follow from the fact that r.h.s of Equation (14) is π_0 for $x = 0$, goes to 1 as $x \rightarrow \infty$ and is strictly increasing in x , while the r.h.s of Equation (15) is π_0 for $x = 0$, goes to 0 as $x \rightarrow \infty$ and is strictly decreasing in x . The definitions of n^* and n^{**} are immediately implied by Corollary 8.13.

□

We note that there are no closed-form solutions for x^* and x^{**} in general. Moreover, we have $n^{**} = n^*$, if we assume symmetry between α_{S_0} and α_{S_1} in the sense of Section 4.5. Finally, note that we can state a similar result for the contrarian case with the mere difference that π^* and π^{**} need to be exchanged.

Under the symmetry assumption for α_{S_0} and α_{S_1} , the problem of calculating P_{n^*} in Lemma 4.6 essentially reduces to a common ruin problem. To see this, note that price updating in the CEU model (Lemma 4.2) immediately implies that the probabilities $P(\{\pi \text{ increases in } t\} | \pi_{t-1}, V)$, $P(\{\pi \text{ decreases in } t\} | \pi_{t-1}, V)$ and $P(\{\pi \text{ remains constant in } t\} | \pi_{t-1}, V)$ remain constant as long as $\pi_{t-1} \in [\pi^{**}, \pi^*]$, where $t \geq 1$ and π^* , π^{**} are from Lemma 8.3. In particular, we have

$$\begin{aligned} P(\{\pi \text{ increases in } t\} | \pi_{t-1}, V) &= P(a_t = \{buy\} | V) =: p_b, \\ P(\{\pi \text{ decreases in } t\} | \pi_{t-1}, V) &= P(a_t = \{sell\} | V) =: p_s, \\ P(\{\pi \text{ remains constant in } t\} | \pi_{t-1}, V) &= P(a_t = \{hold\} | V) =: p_h. \end{aligned} \quad (16)$$

Now, we define for $t \geq 0$

$$i_{t+1} := \begin{cases} i_t + 1 & \text{if } \pi \text{ increases in } t \\ i_t - 1 & \text{if } \pi \text{ decreases in } t \\ i_t & \text{if } \pi \text{ remains constant in } t \end{cases}, \quad (17)$$

where $i_0 = 0$. Then $\pi < \pi^{**}$ is equivalent to $i_t = -n^*$ and $\pi > \pi^*$ is equivalent to $i_t = n^*$ under the symmetry assumption. Let π_{i_t} denote the price process.

We derive the probability of an informational cascade under buy herding given that $V = 0$. The case of sell herding when $V = 1$ is symmetric.

Let A_{ICB}^π denote the event of a buy side informational cascade given that the price is π . Then, initially, we have

$$\begin{aligned} P(A_{ICB}^{\pi_0} | V = 0) &= P(A_{ICB}^{\pi_1} \cap \{i_1 - i_0 = 1\} | V = 0) \\ &\quad + P(A_{ICB}^{\pi_{-1}} \cap \{i_1 - i_0 = -1\} | V = 0) \\ &\quad + P(A_{ICB}^{\pi_0} \cap \{i_1 - i_0 = 0\} | V = 0). \end{aligned} \quad (18)$$

We have decomposed $A_{ICB}^{\pi_0}$ disjointly and then used the additivity of probability measures for disjoint events. Now noting that the events A_{ICB}^π and $\{i_1 - i_0 = -1\}$ are independent and incorporating Equations (16), Equation (18) becomes

$$\begin{aligned} P(A_{ICB}^{\pi_0} | V = 0) &= p_b P(A_{ICB}^{\pi_1} | V = 0) + p_s P(A_{ICB}^{\pi_{-1}} | V = 0) \\ &\quad + p_h P(A_{ICB}^{\pi_0} | V = 0). \end{aligned} \quad (19)$$

Denoting $P(k) := P(A_{ICB}^{\pi_k} | V = 0)$ and solving (19) for $k = 0$, we get

$$(1 - p_h)P(0) = p_b P(1) + p_s P(-1).$$

Since this holds for all integers $k \in [-n^* + 1; -n^* + 1]$ and moreover, $P(n^*) = 1$ and $P(-n^*) =$

0, shifting variables to $j = k + n^*$ yields the following system of linear equations

$$\begin{aligned} (1 - p_h)P(j) &= p_b P(j+1) + p_s P(j-1) \quad \forall j = 1, \dots, 2n^* - 1 \\ \wedge \quad P(0) &= 0 \\ \wedge \quad P(2n^*) &= 1. \end{aligned} \tag{20}$$

Since, on the other hand $1 = p_s + p_b + p_h$, we have that

$$(1 - p_h)P(j) = p_b P(j) + p_s P(j).$$

Equations $j = 1, \dots, 2n^* - 1$ from Equation System (20) are, therefore, equivalent to

$$(P(j+1) - P(j)) \frac{p_b}{p_s} = (P(j) - P(j-1)).$$

Let $K := \frac{p_b}{p_s}$. By backwards induction starting at $j = 2n^* - 1$ we then get

$$P(j) - P(j-1) = K^{2n^*-j} (P(2n^*) - P(2n^* - 1)) \tag{21}$$

for $j = 1, \dots, 2n^* - 1$. Now noting that

$$\begin{aligned} 1 = P(2n^*) - P(0) &= \sum_{j=0}^{2n^*} (P(j+1) - P(j)) \\ &= \sum_{j=0}^{2n^*} K^{2n^*-j} (P(2n^*) - P(2n^* - 1)) \\ &= (1 - P(2n^* - 1)) \sum_{j=0}^{2n^*} K^j \\ &= (1 - P(2n^* - 1)) \frac{1 - K^{2n^*+1}}{1 - K}, \end{aligned}$$

where the last equation holds because $\sum_{j=0}^{2n^*} K^j$ is a geometric sum. Solving this for $P(2n^* - 1)$ yields that

$$P(2n^* - 1) = \frac{K - K^{2n^*+1}}{1 - K^{2n^*+1}}.$$

Now noting that $P(j-1) = (P(j-1) - P(j)) + P(j)$ for $j = 1, \dots, 2n^* - 1$ and inserting

Equations (21) allows us to invoke another backward induction argument to conclude that

$$P(j) = \frac{K^j - K^{2n^*}}{1 - K^{2n^*}}. \quad (22)$$

Setting $j = n^*$ in Equation (22), basic algebra to transform the fraction yields

$$\begin{aligned} P(n^*) &= K^{n^*} \frac{1 - K^{n^*}}{1 - K^{2n^*}} \\ &= K^{n^*} \frac{1}{1 + K^{n^*}} \\ &= \frac{1}{\exp(-\ln(K)n^*) + 1}. \end{aligned}$$

Noting that $p_b = \mu(1 - q) + \theta$ and $p_s = \mu q + \theta$ if $V = 0$, and plugging these quantities in for K yields the formula for P_{n^*} in Equation (7). □

We state an immediate consequence of Lemma 4.6.

Corollary 8.4. *Under the same conditions as in Lemma 4.6, the probability of a correct informational cascade is given by*

$$\tilde{P}_{n^*} = \frac{1}{\exp\left(-\ln\left(\frac{\mu q + \theta}{\mu(1-q) + \theta}\right) n^*\right)}. \quad (23)$$

The prove is identical to the one of Lemma 4.6. It implies, in particular that Equation (22) becomes

$$\tilde{P}(j) = \frac{K^{-j} - K^{-2n^*}}{1 - K^{-2n^*}} \quad (24)$$

for arbitrary $j \in [0; 2n^*]$, where j is the number of price decreases that needs to be observed before the correct informational cascade occurs. □

Proof of Lemma 4.7: To develop an understanding for the idea of the proof note that a parameter shift can have two effects on P_{n^*} . First, it may affect P directly. Second, it may cause n^* to vary. Noting that P_{n^*} decreases in n^* , this has an indirect effect on the probability of wrong herds. Changes in n^* occur as discrete jumps. Locally, this indirect

effect on P is, therefore, zero. It becomes relevant only for larger parameter shifts.

We first summarize the relevant calculus in a support lemma. Note that under the assumptions of Section 4.5, we have that $ask^{CEU} = ask$ if the market is prone to buy herding and buy contrarianism and that $bid^{CEU} = bid$ if the market is prone to sell herding and sell contrarianism. Hence, we can rely on the analytics of ask and bid from the Avery and Zemsky model to derive the results of Lemma 8.5.

Lemma 8.5. *Consider the CEU model. Let P_{n^*} be as in Equation (7). Then the following hold*

- (i) $\frac{\partial P}{\partial \mu} < 0$, $\frac{\partial P}{\partial q} < 0$
- (ii) $\frac{\partial P_x}{\partial x} < 0$
- (iii) $\frac{\partial ask}{\partial \mu} > 0$, $\frac{\partial ask}{\partial q} > 0$
- (iv) $\frac{\partial bid}{\partial \mu} < 0$, $\frac{\partial bid}{\partial q} < 0$
- (v) $\frac{\partial CEUS}{\partial \mu} = 0$
- (vi) $\frac{\partial CEUS}{\partial \delta_0} > 0$ iff $\alpha_S > E[V \mid S, \pi]$;
 $\frac{\partial CEUS}{\partial \delta_0} < 0$ iff $\alpha_S > E[V \mid S, \pi]$
- (vii) $\frac{\partial CEUS}{\partial q} = \frac{\partial E_q[V \mid S, \pi]}{\partial q} + \delta_q \left(\frac{\partial \alpha_q}{\partial q} - \frac{\partial E_q[V \mid S, \pi]}{\partial q} \right) + \frac{\partial \delta_q}{\partial q} (\alpha_q - E_q[V \mid S, \pi])$
- (viii) $\frac{\partial \delta_q}{\partial q} > 0$ if $\delta = \delta_{S_0}$ and $\pi > 0.5$ or if $\delta = \delta_{S_1}$ and $\pi < 0.5$.
- (ix) $\frac{\partial E_q[V \mid S_0, \pi]}{\partial q} < 0$, $\frac{\partial E_q[V \mid S_1, \pi]}{\partial q} > 0$

Proof: For (i) note that

$$\exp \left(-\ln \left(\frac{\mu(1-q) + \theta}{\mu q + \theta} \right) \right) = \frac{\mu q + \theta}{\mu(1-q) + \theta}$$

Differentiating the r.h.s. with respect to μ and q yields quantities > 0 . Applying the quotient rule, therefore implies (i).

For (ii) note that

$$\frac{\partial P_x}{\partial x} = \frac{-\ln \left(\frac{\mu q + \theta}{\mu(1-q) + \theta} \right) \exp(\cdot)}{f^2}.$$

Noting that $\mu q + \theta > \mu(1 - q) + \theta$ implies that $\ln(\cdot) > 0$ and, thus (ii).

For (iii) and (iv) we refer to the reader to the market microstructure literature, e.g. Glosten and Milgrom (1985).

(v) follows from the assumption that α is independent of μ .

For (vi) note that

$$\frac{\partial \delta_{\delta_0}}{\partial \delta_0} = \frac{1}{f^2} > 0.$$

The remainder follows from the definition of CEU_S .

(vii) is a mere application of differentiation rules.

For (viii) note that e.g.

$$\delta_{S_0} = \frac{\delta_0}{(1 - \delta_0) [\pi(1 - q) + q(1 - \pi)] + \delta_0}.$$

Differentiating with respect to q yields that the sign of $\frac{\partial \delta_q}{\partial q}$ is determined by $-\delta_0(1 - \delta_0)(1 - 2\pi)$, which is greater than 0 if and only if $\pi > 0.5$. The argument is symmetric for δ_{S_1} .

Finally, (ix) follows from the literature, see e.g. Chamley (2004). □

For the main proof we make the assumption that the immediate effects of q on n^* dominate the ancillary effects transmitted through changes in the ambiguity parameters. That is, we assume that the increasing effect of q on the ask price and the decreasing effect of q on the additive component of S_0 's CEU belief

$$\frac{\partial ask}{\partial q} - (1 - \delta_q) \frac{\partial E_q[V \mid S_0, \pi]}{\partial q}$$

dominate the ancillary effect that an increase in q actually increases CEU_{S_0} due to its effect on δ and α

$$\delta_q \frac{\partial \alpha_q}{\partial q} + \frac{\partial \delta_q}{\partial q} (\alpha_q - E_q[V \mid S_0, \pi]).$$

That is, the difference of these terms should be positive.

Similarly, the difference between

$$(1 - \delta_q) \frac{\partial E_q[V \mid S_1, \pi]}{\partial q} - \frac{\partial bid}{\partial q}$$

and

$$\delta_q \frac{\partial \alpha_q}{\partial q} + \frac{\partial \delta_q}{\partial q} (E_q[V \mid S_1, \pi] - \alpha_q)$$

should also be positive.

Then, (i)-(v) together imply that an increase in μ decreases P and increases n^* . For the increase in n^* note that an increase in μ increases the ask price and decreases CEU_{S_0} (decreases the bid price and increases CEU_{S_1}). The decrease in P and the increase n^* together imply unambiguously that P_{n^*} decreases in μ .

(i)-(iv) and (vii) in conjunction with our previously made assumptions imply the effects of an increase in q on the probability of a wrong cascade. The assumptions are required to ensure that q unambiguously increases n^* . Beyond that, the argument is identical to the one of the increase of μ .

(vi) implies that an increase in δ_0 causes investors to more heavily rely on their gut feel parameter α . In the case of contrarianism this means that CEU beliefs become less elastic to changes in π , which shifts the cut-off points towards π_0 and, hence decreases n^* , which in turn implies an increase in P_{n^*} . In the case of herding CEU beliefs become more elastic, which again shifts the cut-off points towards π_0 .

For the increase in α note that we define such an increase as follows: α^1 is said to be greater than α^2 if and only if $\alpha^1(\pi) < \alpha^2(\pi) \forall \pi \in (0; \pi_0)$ and $\alpha^1(\pi) > \alpha^2(\pi) \forall \pi \in (\pi_0; 1)$. Now consider some $\pi > \pi_0$. An increase in α then implies an increase in CEU_S regardless of the signal type. For the low signal prone to buy herding this means that the cut-off point π^* moves left, i.e. n^* decreases and P_{n^*} increases. For the high signal, who is prone to contrarianism this means that π^* moves right, i.e. n^* increases and P_{n^*} decreases. The argument is symmetric if $\pi < \pi_0$.

□

Proof of Lemma 5.3: We start by proving that $P_{n^*} \leq P_{WH}^\epsilon$. We can again focus on the buy herding case given $V = 0$ due to symmetry. For ease of notation we assume without loss of generality that $\epsilon = 1$. The line of reasoning is identical if $0 < \epsilon < 1$.

Let i_t be defined as in Equation (17) and let π_{i_t} be the corresponding price process. Moreover, let p_s and p_b be the true buy and sell probabilities in the perturbed model given $V = 0$. Note that p_b and p_s correspond to the respective buy and sell probabilities in the CEU model. Finally, let \tilde{p}_s and \tilde{p}_b be the corresponding buy and sell probabilities as perceived by

the market.

We note that for $i_t \in [-k^* + 1; k^* - 1]$, we have $p_s = \tilde{p}_s$ and $p_b = \tilde{p}_b$ since both informed traders act as in the baseline model without ambiguity.

For $i_t \in [k^*; n^* - 1]$, we have that S_0 holds and S_1 buys and, thus,

$$\frac{p_b}{p_s} = \frac{\mu(1-q) + \theta}{\mu q + \theta} < \frac{\mu(1-q) + \theta}{\theta} = \frac{\tilde{p}_b}{\tilde{p}_s}. \quad (25)$$

Hence, the CEU model assumptions imply a lower probability for strong buy side accumulations than in the perturbed model. This indicates that P_{n^*} underestimates P_{WH}^ϵ .

Correspondingly, for $i_t \in [-n^* + 1; -k^*]$, we have that S_1 holds and S_0 sells, thus yielding

$$\frac{p_b}{p_s} = \frac{\mu(1-q) + \theta}{\mu q + \theta} > \frac{\theta}{\mu q + \theta} = \frac{\tilde{p}_b}{\tilde{p}_s}. \quad (26)$$

Hence, the CEU model assumptions imply a higher probability for strong sell side accumulations than the perturbed model. This indicates that P_{n^*} overestimates P_{WH}^ϵ .

Now, aggregating the net underestimation and net overestimation for some $i_{t,1} \in [-n^* + 1; -k^*]$ and some $i_{t,2} \in [k^*; n^* - 1]$ respectively, we get

$$\begin{aligned} & \left[\frac{\mu(1-q) + \theta}{\mu q + \theta} - \frac{\mu(1-q) + \theta}{\theta} \right] + \left[\frac{\mu(1-q) + \theta}{\mu q + \theta} - \frac{\theta}{\mu q + \theta} \right] \\ &= \mu \frac{\theta(1-2q) - \mu q(1-q)}{\theta(\mu q + \theta)} < 0. \end{aligned}$$

Due to symmetry we can consequently infer that for $i_t \in [-n^* + 1; n^* - 1]$, P_{n^*} in total underestimates P_{WH}^ϵ .

For $i_t \leq -n^*$ P_{n^*} underestimates P_{WH}^ϵ in the sense that in the CEU model social learning stops and the sell herd can never be broken to result in a buy herd after all.

For $i_t \geq n^*$, P_{n^*} overestimates P_{WH}^ϵ in the sense that in the CEU model social learning stops and the buy herd can never be broken to result in a sell herd after all.

Since, however, for $i_t \geq n^*$, we have that

$$\frac{\tilde{p}_s}{\tilde{p}_b} = \frac{\theta}{\mu q + \theta}, \quad (27)$$

and for $i_t \leq -n^*$, we have

$$\frac{\tilde{p}_b}{\tilde{p}_s} = \frac{\theta}{\mu q + \theta}, \quad (28)$$

we conclude that the probability to observe a sell herd given that there is currently buy herding is exactly equal to the probability to observe a buy herd given that there is currently sell herding. Hence, the associated over- and underestimation of P_{n^*} versus P_{WH}^ϵ cancel out.

In total P_{n^*} is, thus, a lower boundary for P_{WH}^ϵ .

We move to show that $P_{k^*} \geq P_{WH}^\epsilon$. For this to see, note that as above the probability of i_t becoming $\geq |k^*|$ in the CEU model and the perturbed CEU model is the same. Again, we focus on the buy herding case, where $V = 0$ and when $\epsilon = 1$ without loss of generality.

In the perturbed model, the probability of observing a price reversal given that $i_t \geq k^*$ is relatively higher than the probability of observing a price reversal into the opposite direction given that $i_t \leq -k^*$. Note that if $i_t \geq k^*$, then $\frac{\tilde{p}_s}{\tilde{p}_b} = \frac{\theta}{\mu(1-q)+\theta}$. Similarly, if $i_t \leq -k^*$, then $\frac{\tilde{p}_b}{\tilde{p}_s} = \frac{\theta}{\mu q + \theta}$. Observing that the second ratio is smaller than the first one, this confirms our claim.

Now, invoking a similar symmetry argument as in the lower boundary case, we conclude that P_{k^*} is, indeed an upper boundary of P_{WH}^ϵ .

□

If $k^* = n^*$, then $P_{WH}^{NCK} = P_{n^*}$. This implies in particular that probabilities of wrong herds and wrong learning coincide if there is no bid-ask spread.

8.C Formal Results for the Perturbed Model

The perturbed CEU model is described by the following two Propositions.

Proposition 8.6. *Equilibrium Prices in the Perturbed CEU Model*

For any time t , let bid_t and ask_t be the bid and ask prices that are quoted in the Avery and Zemsky model. Let π_t be the public belief in the perturbed CEU model. Moreover, let

$$\begin{aligned} bid_t^\epsilon &:= \frac{(\mu(1-q)\epsilon + \theta)\pi_t}{(\mu(1-q)\epsilon + \theta)\pi_t + (\mu q\epsilon + \theta)(1 - \pi_t)}, \\ bid_t^{1-\epsilon} &:= \frac{(\mu(q(1-\epsilon) + (1-q)) + \theta)\pi_t}{(\mu(q(1-\epsilon) + (1-q)) + \theta)\pi_t + (\mu((1-q)(1-\epsilon) + q) + \theta)(1 - \pi_t)}, \\ ask_t^\epsilon &:= \frac{(\mu q\epsilon + \theta)\pi_t}{(\mu q\epsilon + \theta)\pi_t + (\mu(1-q)\epsilon + \theta)(1 - \pi_t)}, \\ ask_t^{1-\epsilon} &:= \frac{(\mu(q + (1-q)(1-\epsilon)) + \theta)\pi_t}{(\mu(q + (1-q)(1-\epsilon)) + \theta)\pi_t + (\mu((1-q) + q(1-\epsilon)) + \theta)(1 - \pi_t)}. \end{aligned}$$

Then, the market maker quotes the following bid and ask prices

$$bid_t^{\epsilon-CEU} = \begin{cases} \min\{bid_t^\epsilon; CEU_{S_0}(\pi_t)\}, & \text{if } CEU_{S_0}(\pi_t) \geq bid_t \\ bid_t, & \text{if } CEU_{S_0}(\pi_t) < bid_t \wedge CEU_{S_1}(\pi_t) \geq bid_t \\ bid_t^{1-\epsilon}, & \text{if } CEU_{S_0}(\pi_t) < bid_t \wedge CEU_{S_1}(\pi_t) < bid_t \end{cases}$$

and

$$ask_t^{\epsilon-CEU} = \begin{cases} \max\{ask_t^\epsilon; CEU_{S_1}(\pi_t)\}, & \text{if } CEU_{S_1}(\pi_t) \leq ask_t \\ ask_t, & \text{if } CEU_{S_1}(\pi_t) > ask_t \wedge CEU_{S_0}(\pi_t) \leq ask_t \\ ask_t^{1-\epsilon}, & \text{if } CEU_{S_1}(\pi_t) > ask_t \wedge CEU_{S_0}(\pi_t) > ask_t. \end{cases}$$

Proof: The proof is essentially a repeated application of Bayes' rule and game theoretic arguments as in the proof of Lemma 4.1. We outline the proof for $bid_t^{\epsilon-CEU}$ when $CEU_{S_0}(\pi_t) \geq bid_t$. The arguments for the other cases are similar.

First note that by monotonicity of the informed traders asset valuation, i.e. $CEU_{S_0}(\pi_t) < CEU_{S_1}(\pi_t)$, the high signal does not sell if the low signal does not sell. Consequently, by Bayes' rule the bid price under the zero-profit condition for the market maker is

$$bid_t^{\epsilon-CEU} = \frac{\tilde{P}(a_t = \{sell\} | H_t, V = 1)P(V = 1 | H_t)}{\tilde{P}(a_t = \{sell\} | H_t)} = bid_t^\epsilon,$$

since $P(V = 1 | H_t) = \pi_t$, $\tilde{P}(a_t = \{sell\} | H_t, V = 1) = \mu(1 - q)\epsilon + \theta$ and $\tilde{P}(a_t = \{sell\} | H_t) = (\mu(1 - q)\epsilon + \theta)\pi_t + (\mu q\epsilon + \theta)(1 - \pi_t)$. Thus, noting that $bid_t^\epsilon > bid_t$, we have to distinguish between the case where $CEU_{S_0}(\pi_t) < bid_t^\epsilon$ and where $CEU_{S_0}(\pi_t) \geq bid_t^\epsilon$.

If $CEU_{S_0}(\pi_t) < bid_t^\epsilon$, then quoting $bid_t^{\epsilon-CEU} = bid_t^\epsilon$ would cause the low signal to sell at π_t . Yet, if S_0 sells, the market maker makes zero-profit only when quoting $bid_t < bid_t^\epsilon$. Hence, he makes an average loss on every sell he fills of $bid_t - bid_t^\epsilon$, causing him to eventually go out of business. As long as the market maker quotes $bid_t^{\epsilon-CEU} > CEU_{S_0}(\pi_t)$ and S_0 sells at π_t a similar argument applies.

Consequently, it must be that $bid_t^{\epsilon-CEU} \leq CEU_{S_0}(\pi_t)$ to ensure that S_0 holds. When quoting $bid_t^{\epsilon-CEU} < CEU_{S_0}(\pi_t)$, then the market maker makes an expected profit of $bid_t^\epsilon - bid_t^{\epsilon-CEU}$. His Bertrand competition can then quote a more competitive bid price, \widetilde{bid} , where

$$bid_t^{\epsilon-CEU} < \widetilde{bid} < CEU_{S_0}(\pi_t).$$

By this the competition draws away all noise traders from the market maker, making a

slightly smaller but still positive expected profit $bid_t^\epsilon - \widetilde{bid}$ on every sell they fill.

The market maker's best response to \widetilde{bid} would then be a similar increase in the quoted bid price. This price war continues until

$$bid_t^{\epsilon-CEU} = \widetilde{bid} = CEU_{S_0}(\pi_t).$$

Hence, for $CEU_{S_0}(\pi_t) < bid_t^\epsilon$, $bid_t^{\epsilon-CEU} = CEU_{S_0}(\pi_t)$ is the equilibrium bid price.

If $CEU_{S_0}(\pi_t) \geq bid_t^\epsilon$, then the market maker quotes $bid_t^{\epsilon-CEU} = bid_t^\epsilon$ according to the zero profit condition. Hence,

$$bid_t^{\epsilon-CEU} = \min\{bid_t^\epsilon; CEU_{S_0}(\pi_t)\},$$

if the low signal stops selling.

□

Key thing to note here is that $ask_t^{\epsilon-CEU} = ask_t$ ($bid_t^{\epsilon-CEU} = bid_t$) as long S_1 buys (S_0) sells. As a consequence, the analytics of a switch from selling to buying and vice versa are the same as in the unperturbed CEU model. This, in turn implies that Theorems 4.3, 4.4 and 4.5 also hold in the perturbed model.

Moreover, since $bid_t^\epsilon < ask_t^{1-\epsilon}$ ($bid_t^{1-\epsilon} < ask_t^\epsilon$), there is a positive bid-ask spread even if all informed traders take the same action. This is because the market (maker) believes there is a share of ϵ SEU traders in the market. Hence, there is no complete informational cascade in the perturbed CEU model since the market still infers information from the trades and updates prices accordingly. This is formalized in

Proposition 8.7. Public Belief Updating in the perturbed CEU Model

Under the assumptions and with the notation of Proposition 8.6, let

$$\begin{aligned} h_t^1 &:= \frac{(\mu(1-q)(1-\epsilon) + \theta)\pi_t}{(\mu(1-q)(1-\epsilon) + \theta)\pi_t + (\mu q(1-\epsilon) + \theta)(1-\pi_t)}, \\ h_t^2 &:= \frac{(\mu q(1-\epsilon) + \theta)\pi_t}{(\mu q(1-\epsilon) + \theta)\pi_t + (\mu(1-q)(1-\epsilon) + \theta)(1-\pi_t)}. \end{aligned}$$

Then:

(1) *If the high CEU-signal buys and the low CEU-signal sells at time t , then π_{t+1} is as in*

the model of Avery and Zemsky, see Equation (39).

(2) If the high CEU-signal buys and the low signal holds at time t , then

$$\pi_{t+1} = \begin{cases} ask_t, & \text{if } a_t = \{buy\} \\ bid_t^c, & \text{if } a_t = \{sell\} \\ h_t^1, & \text{if } a_t = \{hold\}. \end{cases} \quad (29)$$

(3) If the high CEU-signal holds and the low CEU-signal sells at time t , then

$$\pi_{t+1} = \begin{cases} ask_t^\epsilon, & \text{if } a_t = \{buy\} \\ bid_t, & \text{if } a_t = \{sell\} \\ h_t^2, & \text{if } a_t = \{hold\}. \end{cases} \quad (30)$$

(4) If both CEU-signals buy at time t , then

$$\pi_{t+1} = \begin{cases} ask_t^{1-\epsilon}, & \text{if } a_t = \{buy\} \\ bid_t^c, & \text{if } a_t = \{sell\} \\ \pi_t, & \text{if } a_t = \{hold\}. \end{cases} \quad (31)$$

(5) If both CEU-signals sell at time t , then

$$\pi_{t+1} = \begin{cases} ask_t^\epsilon, & \text{if } a_t = \{buy\} \\ bid_t^{1-\epsilon}, & \text{if } a_t = \{sell\} \\ \pi_t, & \text{if } a_t = \{hold\}. \end{cases} \quad (32)$$

Proof: The proof is similar to the one of Lemma 4.2. For the sake of completeness let us consider case (4), which corresponds e.g. to a buy herding regime.

By Bayes' rule the updated prior belief after action a_t is observed is given by

$$\pi_{t+1} = P[V = 1|H_{t+1}] = \frac{\tilde{P}(a_t|H_t, V = 1)P(V = 1|H_t)}{\tilde{P}(a_t|H_t)}.$$

When choosing $a_t = \{buy\}$ and $a_t = \{sell\}$, then the updating rule for π_{t+1} after an observed

buy and sell immediately follow from Proposition 8.6. That the price remains stable after a hold can also be inferred from the above formula. Intuitively, it reflects that the market considers a hold to be conducted by a noise trader for sure. All informed traders that are considered to be present on the market either buy (high signal S_1 or CEU preferences) or sell (low signal S_0 with SEU preferences) the asset. As a consequence, a hold bears no informational value.

□

To prove Propositions 5.1 and 5.2 in the main part, we need the following support propositions

Proposition 8.8. *In the perturbed CEU model, let $\pi^* < \pi_0$ ($\pi^{**} > \pi_0$) be the cut-off prices, such that S_1 sell herds for all $\pi < \pi^*$ (S_0 buy herds for all $\pi > \pi^{**}$).*

*Then $\pi_t^{BH} := (\pi_t | \pi_t > \pi^{**})$ is a sub-martingale with respect to H_t and $\pi_t^{SH} := (\pi_t | \pi_t < \pi^*)$ is a super-martingale with respect to H_t .*

Proof: We only show that π_t^{BH} is a sub-martingale. The proof is symmetric for π_t^{SH} . To show that π_t^{BH} is a sub-martingale with respect to H_t , we need to prove that $E[\pi_{t+1}^{BH} | H_t] \geq \pi_t^{BH}$. Since π_{t+1}^{BH} is bounded by definition other martingale properties follow immediately.

Note that showing $E[\pi_{t+1}^{BH} | H_t] \geq \pi_t^{BH}$ is equivalent to showing

$$E[\pi_{t+1} | H_t, \pi_t > \pi^{**}] \geq \pi_t.$$

Let $p_b := P[a_t = \{buy\} | H_t, \pi_t > \pi^{**}]$, $p_s := P[a_t = \{sell\} | H_t, \pi_t > \pi^{**}]$ and $p_h := P[a_t = \{hold\} | H_t, \pi_t > \pi^{**}]$ be the actual probabilities to observe a buy, sell and hold respectively given that S_0 buy herds. Then from Case (4) of Proposition 8.7, we infer that

$$E[\pi_{t+1} | H_t, \pi_t > \pi^{**}] = p_b ask_t^{1-\epsilon} + p_s bid_t^\epsilon + p_h \pi_t. \quad (33)$$

Now define

$$\tilde{p}_b := (\mu(q + (1 - q)(1 - \epsilon) + \theta)\pi_t + (\mu((1 - q) + q(1 - \epsilon) + \theta)(1 - \pi_t))$$

and

$$\tilde{p}_s := (\mu(1 - q)\epsilon + \theta)\pi_t + (\mu q\epsilon + \theta)(1 - \pi_t)$$

to be the unconditional probabilities of a buy and sell respectively as perceived by market participants. Observe that $p_h = \theta$ and that $p_b = \tilde{p}_b + x$ and $p_s = \tilde{p}_s - x$, where $x = \epsilon\mu((1 - q)\pi_t + (1 - \pi_t)q)$.

Moreover, we set

$$\tilde{p}_b^1 := \mu(q + (1 - q)(1 - \epsilon) + \theta)$$

and

$$\tilde{p}_s^1 := \mu(1 - q)\epsilon + \theta$$

to be the corresponding buy and sell probabilities conditional on $V = 1$.

In line with Proposition 8.6, we can then rewrite the r.h.s. of Equation (33) so that we have

$$\begin{aligned}
E[\pi_{t+1} \mid H_t, \pi_t > \pi^{**}] &= p_b \frac{\tilde{p}_b^1}{\tilde{p}_b} \pi_t + \frac{\tilde{p}_s^1}{\tilde{p}_s} \pi_t + p_h \pi_t \\
&= \left((\tilde{p}_b + x) \frac{\tilde{p}_b^1}{\tilde{p}_b} + (\tilde{p}_s - x) \frac{\tilde{p}_s^1}{\tilde{p}_s} + \theta \right) \pi_t \\
&= \left(\tilde{p}_b^1 + \tilde{p}_s^1 + \theta + x \left(\frac{\tilde{p}_b^1}{\tilde{p}_b} - \frac{\tilde{p}_s^1}{\tilde{p}_s} \right) \right) \pi_t \\
&= \left(1 + x \left(\frac{\tilde{p}_b^1}{\tilde{p}_b} - \frac{\tilde{p}_s^1}{\tilde{p}_s} \right) \right) \pi_t,
\end{aligned}$$

where for the last step observe that $\tilde{p}_b^1 = \mu q + \mu(1-q)(1-\epsilon) + \theta$ and $\tilde{p}_s^1 = \mu(1-q)\epsilon + \theta$ and, thus, $\tilde{p}_b^1 + \tilde{p}_s^1 = \mu + 2\theta$. Since moreover $\mu + 3\theta = 1$ by definition the last equality holds. Based on this, however, showing $E[\pi_{t+1} \mid H_t, \pi_t > \pi^{**}] \geq \pi_t$ is equivalent to showing that $\tilde{p}_b^1 \tilde{p}_s - \tilde{p}_s^1 \tilde{p}_b > 0$. Noting that $\tilde{p}_s = \theta + x$ and that $\tilde{p}_b = \mu + \theta - x$ and plugging in the respective formulae, basic manipulations of the l.h.s. of the last inequality yield

$$\begin{aligned}
&\tilde{p}_b^1 \tilde{p}_s - \tilde{p}_s^1 \tilde{p}_b \\
&= \mu x - \mu^2 \epsilon (1-q) + 2\theta x - 2\mu \epsilon (1-q) \theta \\
&= \mu^2 \epsilon [(1+2\theta \epsilon)((1-q)\pi_t + (1-\pi_t)q - (1-q))] \\
&= \mu^2 \epsilon [(1+2\theta \epsilon)(1-\pi_t)(2q-1)] > 0,
\end{aligned}$$

since $q > 0.5$, which concludes the proof. □

Proposition 8.9. *In the perturbed CEU model, let $\pi^* < \pi_0$ ($\pi^{**} > \pi_0$) be the cut-off prices, such that S_0 is a buy contrarian for all $\pi < \pi^*$ (S_1 is a sell contrarian for all $\pi > \pi^{**}$). Then $\pi_t^{SH} := (\pi_t \mid \pi_t > \pi^{**})$ is a super-martingale with respect to H_t and $\pi_t^{SH} := (\pi_t \mid \pi_t < \pi^*)$ is a sub-martingale with respect to H_t .*

Proof: The proof is point-symmetric to the one for Proposition 8.8. □

8.D Discussion of a Purely Optimistic Market in the CEU Model

We limit our attention to the purely optimistic case since the purely pessimistic case is symmetric. The subsequent analysis holds in the CEU model as well as in the perturbed CEU model, yet we state formal results only for the CEU model.

We start by noting that in a purely optimistic market, the high signal always buys. Since $1 = \alpha > E_{\pi_t}[V \mid S_1, H_t]$ for all $\pi_t \in (0; 1)$, it follows that $CEU_{S_1}(\pi_t) > E_{\pi_t}[V \mid S_1] > ask(\pi_t)$ for all $\pi_t \in (0; 1)$.

For the low signal, one of three cases is possible depending on the primary ambiguity δ_0 . If $\delta_0 > \delta^*$, i.e. if it is high enough, then the purely optimistic low signal always buys, too. If $\delta_0 < \delta^{**}$, i.e. if it is low enough, then S_0 essentially behaves as in the case where $\alpha \in (0; 1)$. If δ_0 is between the two cut-off points, then S_0 will buy at low prices but eventually switch into holding as π_t approaches 1.

Lemma 8.10. *In the CEU model with $\alpha = 1$ and $\delta_0 > 0$, the high signal always buys. For the low signal: $\exists \delta^*, \delta^{**} \in (0; 1)$ with $\delta^* > \delta^{**}$, such that*

- S_0 always buys if and only if $\delta_0 > \delta^*$,
- $\exists \pi^* < 1$ such that S_0 sells $\forall \pi_t \in (\pi^*; 1)$ if and only if $\delta_0 < \delta^{**}$,
- $\exists \pi^{**} < 1$ such that S_0 holds $\forall \pi_t \in (\pi^{**}; 1)$ if and only if $\delta^* \geq \delta_0 \geq \delta^{**}$,

where $\delta^* = \frac{-K_1 + \sqrt{K_1^2 + 4K_2}}{2}$ and $\delta^{**} = \frac{-K_3 + \sqrt{K_3^2 + 4K_4}}{2}$ with

$$\begin{aligned} K_1 &= \frac{2(1-q)(\mu(1-q) + \theta) - q(\mu q + \theta)}{(\mu q + \theta) + q(\mu(1-q) + \theta)} \\ K_2 &= \frac{(1-q)[q(\mu q + \theta) - (1-q)(\mu(1-q) + \theta)]}{q[(\mu q + \theta) + q(\mu(1-q) + \theta)]} \\ K_3 &= \frac{2(1-q)(\mu q + \theta) - q(\mu(1-q) + \theta)}{(\mu(1-q) + \theta) + q(\mu q + \theta)} \\ K_4 &= \frac{(1-q)[q(\mu(1-q) + \theta) - (1-q)(\mu q + \theta)]}{q[(\mu(1-q) + \theta) + q(\mu q + \theta)]} \end{aligned}$$

Proof: For S_1 , there is nothing left to show.

For S_0 , note that by a calculus argument similar to the ones used in the proof of Theorem

4.5, we have that S_0 always buys if and only if

$$\frac{\partial CEU_{S_0}}{\partial \pi_t}(1) < \frac{\partial ask}{\partial \pi_t}(1).$$

To see that this inequality holds if and only if $\delta_0 > \delta^*$, plug in the respective formulas to obtain a quadratic inequality of the form $\delta_0^2 + K_1\delta - K_2 > 0$. Observe that the l.h.s. of this inequality has two roots one which is < 0 . Consequently, for this inequality to hold, δ_0 must be greater than the larger root, which is given by δ^* .

Observe again that the calculus arguments from the proof of Theorem 4.3 yield that S_0 sells the asset when the price is in a neighborhood of 1 if and only if

$$\frac{\partial CEU_{S_0}}{\partial \pi_t}(1) > \frac{\partial ask}{\partial \pi_t}(1).$$

As before, this inequality holds if and only if $\delta_0^2 + K_3\delta - K_4 < 0$. A similar argument as in the previous case yields that for $\delta_0 > 0$, this holds if and only if $\delta_0 < \delta^{**}$.

With both cut-off points given, it follows immediately that S_0 holds in a neighborhood of 1 (but never buys) if and only if $\delta^* \geq \delta_0 \geq \delta^{**}$.

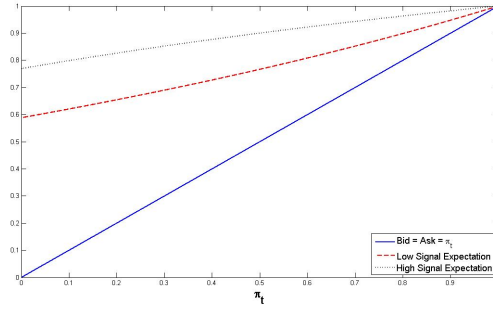
□

Figure 8 illustrates the trade behavior of the CEU traders if they are purely optimistic. In Figure 8 (a), the low signal always buys. Indeed, the primary ambiguity level $\delta_0 = 0.5$ is above the cut-off point $\delta^* = 0.47$ from Lemma 8.10. Since S_1 always buys as well, there is no social learning in a purely optimistic market. The market by default is in the state of an informational cascade.

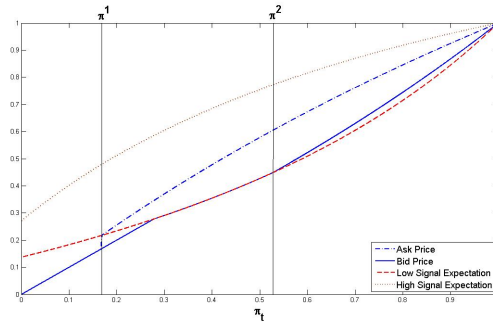
In the perturbed model, the informed traders would drive the asset price towards $\alpha = 1$, regardless of the initial prior and independently of the true value of V .

This is the paragon of irrational exuberance driving the asset price away from its fundamental value. Observe, however, that this is not a case of investor herding in the sense of Definition 3.1. While informed traders accumulate on the buy side of the market, they never change their initial trading decision. The herd-like behavior exhibited by the informed traders is due only to the high degree of perceived ambiguity in conjunction with the fundamentally unrelated pure optimism.

In Figure 8 (b), the low signal sells if $\pi_t > \pi^2$ and buys only if $\pi_t < \pi^1$ due to a low degree of primary ambiguity. Indeed, her additive belief component dominates, making her



(a) High primary ambiguity $\delta_0 = 0.5$



(b) Low primary ambiguity $\delta_0 = 0.1$

Figure 8: Low signal trading decisions in case of pure optimism

Notes: Low signal CEU beliefs as well as bid and ask prices with respect to π_t for high and low primary ambiguity δ_0 . The degree of optimism is $\alpha = 1$, the informed trader share is $\mu = 0.3$ and the signal precision $q = 0.7$. The cut-off points according to Lemma 8.10 are $\delta^* = 0.47$ and $\delta^{**} = 0.22$.

act as though in the baseline model if prices are high. This case is covered by Theorem 4.3. Once the price drops below π^1 a buy contrarian cascade occurs. This happens with positive probability even if $V = 1$. So even a purely optimistic market does not necessarily become confident regarding $V = 1$. Contrary to the case where $\alpha < 1$, there is, however, at least the possibility that the market confidently learns about $V = 1$, since high signal traders always buy.

In the perturbed model, there is no informational cascade. If $V = 0$ is the true state, the majority of low signals in the market will prevent the price from remaining above π^2 for extended periods of time. If the price is below π^2 or even below π^1 , however, the likelihood of observing a price increase is greater than that of a further price decrease. Hence, we would assume, the price to always revert towards π_2 . If $V = 1$, the market will confidently learn

about the correct true state, as the majority of the high signals eventually pushes the price arbitrarily close to 1.

8.E Collection of Additional Results from Avery and Zemsky

For the reader's convenience we have collected relevant formulas for the Avery and Zemsky (1998) model in the following.

Lemma 8.11. *Formulas of the Avery and Zemsky Framework*

In the Avery and Zemsky (1998) framework with initial prior π_0 , informed trader share μ , symmetric binary signals $P[S|V]$ with signal precision q and noise traders that buy, sell or hold with equal probability θ , the following equations hold.

(i) *Buy and sell probabilities conditional on V :*

$$\begin{aligned} P(a_t = \{\text{buy}\} | V = 0) &= P(a_t = \{\text{sell}\} | V = 1) = \mu(1 - q) + \theta \\ P(a_t = \{\text{sell}\} | V = 0) &= P(a_t = \{\text{buy}\} | V = 1) = \mu q + \theta \end{aligned} \quad (34)$$

(ii) *Ask price in t :*

$$\text{ask}_t = \frac{(\mu q + \theta)\pi_t}{(\mu q + \theta)\pi_t + (\mu(1 - q) + \theta)(1 - \pi_t)} \quad (35)$$

(iii) *Bid price in t :*

$$\text{bid}_t = \frac{(\mu(1 - q) + \theta)\pi_t}{(\mu(1 - q) + \theta)\pi_t + (\mu q + \theta)(1 - \pi_t)} \quad (36)$$

(iv) *Expected value asset valuation by low signal in t :*

$$E[V | S_0, H_t] = \frac{(1 - q)\pi_t}{(1 - q)\pi_t + q(1 - \pi_t)} \quad (37)$$

(v) *Expected value asset valuation by high signal in t :*

$$E[V | S_1, H_t] = \frac{q\pi_t}{q\pi_t + (1 - q)(1 - \pi_t)} \quad (38)$$

(vi) Price updating after observing trade action in t :

$$\pi_{t+1} = \begin{cases} ask_t, & \text{if } a_t = \{buy\} \\ bid_t, & \text{if } a_t = \{sell\} \\ \pi_t, & \text{if } a_t = \{hold\} \end{cases} \quad (39)$$

The proofs can be found in Avery and Zemsky (1998).

In addition, we provide some interesting properties of the public belief π_t in the Avery and Zemsky model. The following proposition shows that the public belief can be uniquely identified with any order imbalance in the trade history H_t .

Proposition 8.12. Public Belief And Order Imbalance

Let (μ, q, π_0) be some model parameterization of the Avery and Zemsky (1998) and let noise traders buy, sell or hold with equal probability θ . Moreover, let H_t be some trade history containing b buys, s sells and h holds, where $z := b - s$ denotes the trade imbalance. Then, π_t does not depend on the order at which traders arrive at the market if b, s, h remain unchanged. In particular, π_t only depends on the model parameters and z .

Before we provide the proof we would like to state a few implications of Proposition 8.12. Indeed, it suggests that we can view π_t as a measure for general market sentiment. The larger π_t , the larger the buy side accumulation of traders, the more optimistic the market as a whole and vice versa.

Based on Proposition 8.12, we can also derive the following formula for π_t based on model parameters μ, q and π_0 as well as the order imbalance z .

Corollary 8.13. *With the same notation as in Proposition 8.12, we have*

$$\pi_t(z) = \begin{cases} \frac{(\mu q + \theta)^z \pi_0}{(\mu q + \theta)^z \pi_0 + (\mu(1-q) + \theta)^z (1 - \pi_0)}, & \text{if } z > 0 \\ \frac{(\mu(1-q) + \theta)^{-z} \pi_0}{(\mu(1-q) + \theta)^{-z} \pi_0 + (\mu q + \theta)^{-z} (1 - \pi_0)}, & \text{if } z < 0 \\ \pi_0, & \text{if } z = 0 \end{cases} \quad (40)$$

Proof of Proposition 8.12 and Corollary 8.13 To prove Proposition 8.12, we first show the following

Lemma 8.14. *In the Avery and Zemsky (1998) framework with prior probability π_0 , at any time $\tau \in [1; T]$, we have*

$$\pi_t = P(V = 1|H_t) = \frac{\prod_{t=1}^{\tau} P(a_t|V = 1)\pi_0}{\prod_{t=1}^{\tau} P(a_t|V = 1)\pi_0 + \prod_{t=1}^{\tau} P(a_t|V = 0)(1 - \pi_0)} \quad (41)$$

Proof: We show this via induction over τ . Let $\tau = 1$. Using Equation (39), Bayes' rule and the law of total probability readily imply that

$$\pi_1 = P(V = 1|H_1) = \frac{P(a_1|V = 1)\pi_0}{P(a_1|V = 1)\pi_0 + P(a_1|V = 0)(1 - \pi_0)}.$$

Now let us assume that the statement from Lemma 8.14 is true for any $\tau \geq 1$, then as for $\tau = 1$, we get

$$\pi_{\tau+1} = \frac{P(a_{\tau+1}|V = 1)\pi_{\tau}}{P(a_{\tau+1}|V = 1)\pi_{\tau} + P(a_{\tau+1}|V = 0)(1 - \pi_{\tau})}.$$

Now using the induction assumption, we can plug in the r.h.s. of Equation (41) for π_{τ} and get:

$$\pi_{\tau+1} = \frac{P(a_{\tau+1}|V = 1) \prod_{t=1}^{\tau} P(a_t|V = 1)\pi_0}{C},$$

where

$$\begin{aligned} C := & P(a_{\tau+1}|V = 1) \prod_{t=1}^{\tau} P(a_t|V = 1)\pi_0 \\ & + P(a_{\tau+1}|V = 0) \prod_{t=1}^{\tau} P(a_t|V = 0)(1 - \pi_0). \end{aligned}$$

As we absorb the terms $P(a_{\tau+1}|\cdot)$ into the respective products, we have shown that Equation (41) holds for $\tau + 1$, which concludes the proof. \square

Now continuing the proof of Proposition 8.12, we consider any history H_t of length ≥ 2 (if H_t contains less than two actions, then there is nothing to show). Now let $\sigma(H_t)$ denote an arbitrary permutation of actions contained in H_t , then $\sigma(H_t) =: \tilde{H}_t$ defines a second history with equal length as well as equal number of buys, sells and holds as H_t . Applying Equation 41, we see that $P(V = 1|H_t)$ and $P(V = 1|\tilde{H}_t)$ are identical up to a commutation within the product terms $\prod(\cdot)$. Consequently, we have $P(V = 1|H_t) = P(V|\tilde{H}_t)$, which proves that π_t does not depend on the order of arrival of traders as long as their trading decisions remain unchanged.

For the second part of Proposition 8.12 assume that history H_t contains b buys s sells and h holds. Let us assume without loss of generality that $b \geq s$. Moreover, for notational convenience, let us denote $p^B = P(a_t = \{buy\} | V = \cdot)$, $p^S = P(a_t = \{sell\} | V = \cdot)$ and $p^H = P(a_t = \{hold\} | V = \cdot)$ for the remainder of the proof. Since the order of the actions is not important, we can rewrite Equation (41) as

$$\pi_t = P(V = 1 | H_t) = \frac{(p_1^B)^b (p_1^S)^s (p_1^H)^h \pi_0}{(p_1^B)^b (p_1^S)^s (p_1^H)^h \pi_0 + (p_0^B)^b (p_0^S)^s (p_0^H)^h (1 - \pi_0)}.$$

Now noting that $p_0^S = p_1^B$ and vice versa and that $p_0^H = p_1^H$, we can factorize the denominator so that we get

$$\pi_t = P(V = 1 | H_t) = \frac{(p_1^B)^b (p_1^S)^s (p_1^H)^h \pi_0}{(p_1^B)^s (p_1^S)^s (p_1^H)^h [(p_1^B)^{b-s} \pi_0 + (p_0^B)^{b-s} (1 - \pi_0)]}.$$

Setting $b - s = z$ and reducing the fraction, we get

$$\pi_t = P(V = 1 | H_t) = \frac{(p_1^B)^z \pi_0}{(p_1^B)^z \pi_0 + (p_0^B)^z (1 - \pi_0)}. \quad (42)$$

For $s > b$, symmetry implies that we can simply replace the buy probabilities in Equation (42) with the corresponding sell probabilities, which concludes the proof that π_t only depends on z and the model parameters. To see that Corollary 8.13 holds, use the formulas for $P(a_t | V)$ according to Lemma 8.11.

□

8.F Informational Cascades

Proposition 8.15. *In the two-state, two-trader version of the Avery and Zemsky (1998) framework, an informational cascade occurs if and only if all informed traders take the same action.*

The “if” part of the result generalizes to any number and even to a continuum of states and different informed traders as long as all model parameters are common knowledge and the conditional signal distribution $P(S|V)$ is not constant in V . In general settings, e.g. confounded learning, informational cascades may occur if agents take different actions, compare Exercise 4.6 in Chamley (2004).

Proof:

“ \Leftarrow ”: Let us assume all informed traders take the same action at t . Let this action without loss of generality be a buy. Then:

$$\begin{aligned} & P(a_t = \text{buy} | H_t, \text{“trade is informed”}) \\ &= P(S = S_0 | H_t) + P(S = S_1 | H_t) = 1 \end{aligned} \tag{43}$$

and

$$\begin{aligned} & P(a_t = \text{buy} | H_t, V, \text{“trade is informed”}) \\ &= P(S = S_0 | H_t, V) + P(S = S_1 | H_t, V) = 1 \end{aligned} \tag{44}$$

as well. We also note that the corresponding conditional probabilities for any informed trader action other than a buy are zero. Hence, we get

$$\begin{aligned} & P(a_t = \text{buy} | H_t, V) \\ = & P(\text{“informed buy”} | H_t, V) + P(\text{“uninformed buy”} | H_t, V) \\ = & P(a_t = \text{buy} | H_t, V, \text{“trade is informed”})P(\text{“trade is informed”} | H_t, V) \\ & + P(\text{“uninformed buy”} | H_t, V). \end{aligned}$$

Now noting that the probability μ that a trade is informed and the probability θ that an uninformed trader buys do not depend on the state of the world and applying equations (43) and (44), we get

$$\begin{aligned} & P(a_t = \text{buy} | H_t, V) \\ = & P(a_t = \text{buy} | H_t, \text{“trade is informed”})P(\text{“trade is informed”} | H_t) \\ & + P(\text{“uninformed buy”} | H_t) \\ = & P(a_t = \text{buy} | H_t). \end{aligned}$$

For any action other than buy, we have that $P(a_t | H_t, V) = \theta = P(a_t | H_t)$ and, therefore, the probability is independent of the state of the world, which concludes this part of the proof. We note that common knowledge is crucial to the proof since Equations (43) and (44) would not necessarily hold if the informed traders’ actions were obscured by some unobservable preference parameters.

“ \Rightarrow ”: We proof this indirectly by assuming that without loss of generality the low signal sells at t while the high signal buys. Then:

$$\begin{aligned} & P(a_t = \text{buy} | H_t, \text{“trade is informed”}) \\ &= P(S = S_1 | H_t) = \pi_t q + (1 - \pi_t)(1 - q) \end{aligned} \tag{45}$$

and

$$\begin{aligned} P(a_t = \text{buy} | H_t, V, \text{“trade is informed”}) &= P(S = S_1 | H_t, V) \\ &= \begin{cases} \pi_t q, & V = V_1 \\ (1 - \pi_t)(1 - q) & V = V_0 \end{cases}. \end{aligned} \tag{46}$$

We infer from equations 45 and 46 that

$$P(a_t = \text{buy} | H_t, \text{“trade is informed”}) \neq P(a_t = \text{buy} | H_t, V, \text{“trade is informed”}),$$

which readily implies that $P(a_t = \text{buy} | H_t, V) \neq P(a_t = \text{buy} | H_t)$ and, therefore, concludes the second part of the proof. □

8.G CEU and NEO-Additivity Toolbox

This section is a summary of the most important concepts and results of Chateauneuf et al. (2007) and Eichberger et al. (2010). It is the mathematical foundation for Section 3.

Capacities and Choquet Expected Utility (CEU): Let $S \subset \mathbb{R}$ denote a non-empty set of possible states of the world. Let $\sigma(S) =: \mathcal{E}$ denote the corresponding Borel Sigma-Algebra of all possible subsets of S . Note that by definition $\forall s \in S : \{s\} \in \mathcal{E}$.

Definition 8.16. *Capacity*

A **capacity** is a mapping $\nu : \mathcal{E} \rightarrow [0; 1]$ that assigns likelihood values to events in a way that it fulfills the following properties:

- (i) *Monotonicity:* $\forall E, F \in \mathcal{E}$, where $E \subseteq F : \nu(E) \leq \nu(F)$

(ii) *Normalization*: $\nu(\emptyset) = 0$ and $\nu(S) = 1$.

We note that a capacity defines a normalized measure. A special case of capacities are probability measures. Yet, capacities in general are not additive with respect to \mathcal{E} . This non-additivity implies in particular that for some event E , where $0 < \nu(E) < 1$, we do not necessarily have that $\nu(E) = 1 - \nu(E^C)$, where E^C denotes the complement of E . Therefore, capacities are suited to model agent behavior under ambiguity.

Capacities are designed to explain the Ellsberg paradox of Ellsberg (1961). In Ellsberg's experiment individuals are confronted with the choice of drawing a ball from one of two urns. They know that the first urn contains 50 white balls and 50 black balls while the composition of the second urn is unknown. Subjects win money, if they draw a white ball. Most participants choose to draw from urn 1, where the composition is known. This implies that they assign a probability of less than 50% of drawing a white ball from urn 2. Now, Savage's sure thing principle would predict that when the winning condition is changed to drawing a black ball, subjects should prefer urn 2 to urn 1. A corresponding repetition of the experiment shows, however, that subjects still tend to prefer urn 1 to urn 2, thus, violating Savage's SEU framework.

Next we define the Choquet integral with respect to capacities for a set of simple functions:

Definition 8.17. *Choquet Integral*

Let $f : S \rightarrow B \subset \mathbb{R}$, where B has a finite number of elements. The Choquet integral with respect to the capacity ν is defined as

$$\int f d\nu := \sum_{t \in f(S)} t \cdot [\nu(\{s \mid f(s) \geq t\}) - \nu(\{s \mid f(s) > t\})].^{55}$$

The Choquet integral is interpreted as the expected value under ambiguity. If we think of f being a utility function, it is natural to denote $CEU := \int f d\nu$ as the Choquet Expected Utility of an individual that perceives ambiguity and has ambiguity preferences that are captured by ν .

⁵⁵Note that the term $\nu(\{s \mid f(s) \geq t\}) - \nu(\{s \mid f(s) > t\})$ very much reminds us of decision weights from prospect theory according to Tversky and Kahneman (1992). Yet, while probability weightings are merely distortions of objective probabilities designed to capture individuals' tendencies to wrongly assess given probabilities, capacities model how individuals assign likelihoods to outcomes, for which no (single) probability is available.

NEO-Additive Capacities: We provide a simplified definition for neo-additive capacities that is sufficient for this study.

Definition 8.18. NEO-Additive Capacity

Let π be a probability measure on (S, \mathcal{E}) and let

$$\eta_\alpha = \begin{cases} 0 & E = \emptyset \\ \alpha & E \neq \emptyset \wedge E \neq S \\ 1 & E = S \end{cases}$$

be the Hurwicz capacity and let $\delta, \alpha \in [0; 1]$, then a neo-additive capacity $\nu(\cdot \mid \pi, \delta, \alpha)$ is defined as

$$\nu(E \mid \pi, \delta, \alpha) := (1 - \delta)\pi(E) + \delta\eta_\alpha(E).$$

The CEU with respect to a neo-additive capacity is shown by Chateauneuf et al. (2007) to be

$$\begin{aligned} CEU_{neo}[f] = & (1 - \delta)E_\pi[f] + \delta(\alpha \cdot \max_{x \in B}\{f^{-1}(x)\} \\ & + (1 - \alpha) \cdot \min_{x \in B}\{f^{-1}(x)\}). \end{aligned} \tag{47}$$

The function f again is a simple function in the sense of Definition 8.17. When assuming that informed traders have neo-additive CEU preferences in the Avery and Zemsky (1998) framework, f is the identity as we maintain the assumption of risk neutrality. Since there are only two states $V_0 = 0$ and $V_1 = 1$, the non-additive part simplifies to $\delta(\alpha \cdot 1 + (1 - \alpha) \cdot 0) = \delta\alpha$. The parameter δ describes the degree of perceived ambiguity, while the parameter α measures the attitude towards ambiguity.

The absolute ambiguity attitude in the sense Ghirardato and Marinacci (2002) for individuals with neo-additive CEU preferences is then given by the following

Proposition 8.19. Absolute Ambiguity Attitude

Let \succ_{neo} denote a preference relation that can be represented by a neo-additive capacity $\nu(E \mid \pi, \delta, \alpha)$. Then, \succ_{neo} is ambiguity averse (loving) in the sense of Ghirardato and Marinacci (2002) if and only if $\alpha < (>)E_\pi[\cdot]$. It is ambiguity neutral if and only if $\alpha = E_\pi[\cdot]$.

Proof: According to Proposition 15 in Ghirardato and Marinacci (2002), a preference re-

lation is ambiguity neutral if and only if it is SEU. In the case of neo-additive capacities this would mean that $CEU_{neo} = E_\pi$. Now, let us assume that without loss of generality utilities are normalized (or canonical), that is, $\max\{u^{-1}(x)\}$ from Equation (47) equals 1 and $\min\{u^{-1}(x)\} = 0$. Now solving Equation (47) for α , we get $\alpha = E_\pi$. Hence, neo-additive preferences are ambiguity neutral if and only if $\alpha = E_\pi$.

If $\tilde{\alpha} > E_\pi$, it follows that $CEU_{neo}^{\tilde{\alpha}} > CEU_{neo}^\alpha$. This, in turn, implies that CEU_{neo}^α is more ambiguity averse than $CEU_{neo}^{\tilde{\alpha}}$ according to Definition 4 in Ghirardato and Marinacci (2002). Since we have already shown that CEU_{neo}^α is SEU, Definition 9 in Ghirardato and Marinacci (2002) implies that $CEU_{neo}^{\tilde{\alpha}}$ is ambiguity loving. The argument for absolute ambiguity aversion is symmetric. \square

General Bayesian Updating (GBU) Rule: The following GBU rule for neo-additive capacities is derived and discussed by Eichberger et al. (2010).

Proposition 8.20. General Bayesian Updating

Let $E \subseteq S$ be some conditioning event and let $\nu(\cdot \mid \pi, \delta, \alpha)$ be an unconditional neo-additive capacity. Let $\pi(E) > 0$. Then:

- The capacity $\nu_E(\cdot \mid \pi, \delta, \alpha)$ that is conditioned on E is neo-additive as well;
- The additive probability π is updated to π_E according to Bayes' rule, i.e. $\pi_E(A) = \pi(A \cap E)/\pi(E)$ for $A \in \mathcal{E}$;
- $\alpha_E = \alpha$;
- $\delta_E = \frac{\delta}{(1-\delta)\pi(E)+\delta}$.

8.H Inconsistencies of GBU in the CEU Model

An important reason why α should vary with π is that it allows consistent assumptions regarding asymptotic ambiguity attitudes as the market becomes confident about either state. If $\alpha \in (0; 1)$ is constant, then for both informed trader types

$$\alpha_t^{rel} = \frac{\alpha}{E_{\pi_t}[V \mid H_t, S]} \rightarrow \alpha < 1, \text{ as } \pi_t \rightarrow 1 \quad (48)$$

and

$$\alpha_t^{rel} = \frac{\alpha}{E_{\pi_t}[V | H_t, S]} \rightarrow \infty, \text{ as } \pi_t \rightarrow 0. \quad (49)$$

In other words, informed traders become pessimistic as the market becomes confident about the *high state*. Similarly, traders become absolute optimistic as the market becomes confident about the *low state*. Hence, a fixed α in the CEU model does not guarantee that the informed traders' preference for ambiguity is invariant. Indeed, if an informed trader is optimistic at $t = 0$ and the market confidently learns that the high state is true, the informed trader will eventually become pessimistic. Yet, why of all times, would traders become pessimistic, when the market expresses strong or even full confidence about the high state and vice versa? If the market gets confident about either state of the world, *risk* becomes vanishingly small. In the limiting case that $\pi \in \{0; 1\}$, there is no uncertainty, Knightian or otherwise. Hence, CEU-investors should value the asset at 0 or 1 respectively just like their SEU counterparts from the baseline model. This is guaranteed if $\alpha(0) = 0$ and $\alpha(1) = 1$.

One might argue that there should, indeed, be no perceived ambiguity in the case of full confidence. That is, the degree of perceived ambiguity δ should go $\rightarrow 0$ as $\pi_t \rightarrow \{0; 1\}$. We would agree that such an assumption would be feasible as well but it would be an altogether different model. The way we understand ambiguity in this paper is that it cannot be learned away. The level of primary ambiguity δ_0 , for instance, is associated with the complexity of the financial product or the level of expertise of the trader. Therefore, it does not vanish, even if the market becomes confident about the true state of V . Moreover, the ambiguity stemming from the informed trader's private information is highest when it contrasts to the view of the market, see Figure 2. For a low signal, the perceived informational ambiguity is, indeed, highest if the market confidently believes that $V = 1$. Hence, there is ambiguity if the market is confident, but it's effect on the informed traders' decision making should become marginal. If we assume regularity of α in π , this also implies that neo-additive Choquet preferences in the CEU model are consistent with smooth ambiguity preferences in the Klibanoff et al. (2005) approach.

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